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شُكر وتقدير

شكر وتقدير

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الملخص العربي

عن نظرية الانحناء للسطوح المسطرة في الفضاء الإقليدي ثلاثي الأبعاد

تعتبر الهندسة من أقدم فروع الرياضيات نشأة، فقد عرف الإنسان في فترة ما قبل الميلاد الهندسة كمحسوسات ملموسة وبديهية من الحياة اليومية في المناطق التي يعيش فيها، وقد مرت الهندسة تاريخياً بثلاث مراحل مهمة هي الهندسة القديمة وكان من أشهر النماذج لها هندسة إقليدس التي قام إقليدس بتأصيلها من خلال نظام المسلمات في العهود الإغريقية والتي عالج هلبرت القصور فيها بعد ذلك. والهندسة التحليلية التي أنشأها ديكارت وفيرما ظهرت نتيجة لتقدم الجبر. والهندسة التفاضلية التي نشأت كضرورة لمعرفة سلوك النقطة في المنطقة المتناهية الصغر حولها، ثم بعد ذلك باستخدام المفاهيم المحلية (locally) تطورت وأصبحت تهتم بدراسة الأشكال دراسة موسعة وشاملة (globally).

أي أن الهندسة في البداية كانت أشكالاً وقياسات ومع ظهور الجبر والتوبولوجي أصبحت الهندسة تتناول الوصف وتفسير الظواهر، وليس بالضرورة القياس الذي يترتب عليه اتخاذ القرارات. وأصبح الفضاء هو تجمعاً من الأشياء التي تحقق خواص معينة وتربطها علاقات متبادلة.

وتنقسم الهندسة التفاضلية إلى دراسات كلاسيكية ودراسات حديثة، وتعتبر هندسة المستقيما ت (line geometry) أحد موضوعات الهندسة التفاضلية الكلاسيكية التي بدأ ظهورها في النصف الأخير من القرن التاسع عشر، وأصبحت الآن من أحدث الموضوعات في الهندسة لما لها من تطبيقات في حياتنا العملية.

قُدمت هذه الدراسة من قبل العالم بولكر (Plücker [32]) حيث أصدر كتاب في هذا الموضوع، ورياضيون آخرون أمثال كلاين (Klein [22]) وهالافاتي (Hlavaty [17]) كما أنه يوجد عدد كبير من الدراسات المتعلقة بهذا الموضوع وتطبيقاته ومنها على سبيل المثال بلاشكا (Blaschke [5]) وبوتيما وروث (Bottema and Roth [6])، كذلك قام الباحثان بوتمان ووالنير(Pottman and Wallner [33]) بتطوير الخصائص الهندسية لهندسة الخطوط المستقيمة والتي أصبحت الأساس لوصف كينماتيكا (علم الحركة) الهندسة التفاضلية المولدة بالسطوح المسطرة (Ruled surfaces).

تعتبر دراسة الهندسة التفاضلية للسطوح المسطرة (Ruled surfaces) والمولدة بخط مستقيم في الفضاء الإقليدي ثلاثي الأبعاد من أهم فروع الهندسة التفاضلية وذلك نتيجة لإرتباطها المباشر بالعديد من العلوم الهندسية والفيزيائية وغير ذلك من النواحي التطبيقية، أنظر على سبيل المثال المراجع [24, 33, 37-40].

من ناحية أخرى في الفترة (1845م-1875م) عرّف وليم كلايفورد (W.Clifford [7]) الأعداد الإزدواجية (dual numbers). وبعد ذلك استخدم شتودي (E. Study) الأعداد الإزدواجية والمتجهات الإزدواجية (dual numbers and dual vectors) في دراساتهِ عن الهندسة الحركية للخط المستقيم. وكرّس اهتمامه لتمثيل الخطوط المستقيمة بواسطة المتجهات الإزدواجية. حيث في عام (1903م) استحدث (E. Study map [42]) التحويل الآتي:

" مجموعة الخطوط المستقيمة في الفضاء الإقليدي E^3 تكون في حالة تناظر أحادي مع مجموعة النقاط على كرة الوحدة الإزدواجية (dual unit sphere)".

باستخدام مبدأ شتودي للتحويل يمكننا اعتبار دراسة هندسة الخطوط المستقيمة في الفضاء الإقليدي الثلاثي E^3 على أنها هندسة النقاط على سطح كرة الوحدة الإزدواجية، بينما دراسة هندسة السطوح المسطرة في الفضاء الإقليدي الثلاثي E^3 على أنها هندسة المنحنيات على سطح كرة الوحدة الإزدواجية. ومن الدراسات الحديثة في هذا الموضوع، أنظر المراجع [1-3, 25, 33].

وفي هذه الرسالة اعتماداً على مبدأ شتودي للتحويل سوف نقدم نظرية الانحناء للسطوح في الفضاء الإقليدي ثلاثي الأبعاد. وتنقسم هذه الرسالة إلى أربعة فصول تتضمن ما حصلنا عليه من نتائج كما يلي:

في الفصل الأول تم عرض بعض المبادئ والمفاهيم الأساسية في علم الهندسة التفاضلية والتي تلزم لإتمام الرسالة. حيث قدمنا أولاً بصورة موجزة بعض المفاهيم المتعلقة بالمنحنيات في الفضاء الإقليدي الثلاثي E^3، وعرضنا مفهوم بدائل بيرتراند للمنحنيات (Bertrand offsets) وأثبتنا نظريات توضح خواصها. وثانياً عرضنا الهندسة التفاضلية للخط المستقيم باستخدام إحداثيات بولكر (Plucker coordinates). وأخيراً تم توضيح مفهوم الأعداد والمتجهات الإزدواجية (dual numbers and dual vectors)، وتم تقديم تحويل شتودي تفصيلياً.

في الفصل الثاني تم تعريف السطح المسطر ومعادلته البارامترية، وعرضنا إطار بلاشكا الكلاسيكي ودوال الانحناء، ومن ثم قدمنا دراسة مستفيضة عن الخواص الهندسية للسطوح المسطرة القابلة للإنبساط. أخيراً عرضنا إطار بلاشكا ودوال الانحناء للسطح المسطر بالصورة الإزدواجية. وتم إعطاء مثال على ذلك وحالات خاصة.

في الفصل الثالث عرضنا بعض العلاقات الهندسية بين المنحنيات والسطوح المسطرة وذلك من خلال حلول نظام من المعادلات التفاضلية الخطية، وأوجدنا عديد من الشروط الخاصة لمنحنى فضائي وكذلك منحنى واقع على سطح.

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On the Curvature Theory of Ruled Surfaces In Euclidean 3-Space

A Thesis

Submitted to the Mathematic Department in
Partial Fulfillment of the Requirements for The Master's
Degree. In Pure Mathematics (Differential Geometry)

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ABSTRACT

ABSTRACT

An oriented line in a moving rigid body generates a ruled surface in Euclidean 3-Space E^3. The ruled surface is called the line trajectory of the moving body. In this dissertation, the primary interest is to develop the curvature theory of ruled surfaces in Euclidean 3-Space E^3.

In 1903 [42] E. Study introduced the following map:
" The set of oriented lines in the Euclidean 3-space E^3 is in one-to-one correspondence with the set of points on the dual unit sphere".

By using the principle of E. Study we can consider the study geometry of the ruled surface in the three dimensional Euclidean space E^3 as the geometry of curve on the surface of the dual unit sphere.

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ARABIC SUMMARY

Introduction

The theory of line geometry was first introduced by Plücker [32] who published a book on the subject. Other mathematicians, including Klein [22], Hlavaty [17], and Pottman, H. & Wallner, J. [33] developed the differential properties of line geometry that became the basis for describing the differential geometry of the kinematically generated ruled surfaces. The differential line geometry is continuously developing, with some of the most influential works, see for example [1-4, 11-15].

The application of line geometry and dual number representation of line trajectories has been developed by Bottema & Roth [6], and Blaschke, W. [5]. A more recent description of this representation can be found in the works [36-43]. The dual number is used to recast the point displacement relationship into relationships of lines. The dual numbers were first introduced by Clifford [7] after him Study [42] used it as a tool for his research on the differential line geometry. There exists vast literatures on this subject for example: Blaschke [5], Karger, A. & Novak, J. [21], Veldkamp [44], and Pottman, H. & Wallner, J. [33].

In this work the emphasis is to develop the curvature theory of ruled surfaces using E. Study's map. Using dual vector algebra and calculus, we may develop a three-dimensional theory by dualizing the results of spherical curves. The geometric properties of the ruled surface are presented in terms of dual curvature functions. This thesis consists of four chapters. A detailed description of these chapters would run as follows:

In chapter 1, firstly we shown some principles and concepts in the differentiation geometry that is necessary to complete the thesis, we have first presented it in a brief picture some of concepts related to the curves in the three dimensional Euclidean space E^3. We also presented the concepts of Betrand offsets of curves and we proved theories show these properties. Secondly we presented the differentiation geometry for the straight line by using the Plücker coordinates. Finally, we have clarified the concept of dual numbers vectors, and presented E. Study's map in details.

In chapter 2, we introduced the ruled surface and its parametric equation. Also, we have presented the classical Blaschke frame and curvature functions. Further, we presented a profound study about the characters of geometry in the

developable ruled surfaces. Finally, we presented the dual Blaschke frame and dual curvature functions of ruled surface, as well as an example is presented and discussed in detail.

In chapter 3, the relationship between the curves and the ruled surfaces by giving solutions to the system of linearly differential equations has been introduced. Moreover, many geometrical conditions related with the kind and positions of the space curve with the ruled surface are established.

In chapter 4, we presented Betrand offsets of ruled surface and curvature functions in the scalar formulation, we also presented its dual formulation and proved theories appear corresponding of properties. Moreover, an example of application has been given and investigated in detail.

We have provided this thesis with a list of references which we have benefited from in this thesis. We have included also a list of academic expressions used with its interpretation.

CHAPTER 1

Preliminaries

In the Euclidean three space E^3, the curvature properties of Bertrand space curves which are presented will be used to develop curvature theory in the later chapters for ruled surfaces. Latter on, dual numbers, vectors and E. Study's map are formulated.

(1.1) Space Curves in Euclidean 3-space

A space curve in Euclidean 3-space E^3 is a continuous mapping of an open interval of a real parameter $t \in I \subset \mathbb{R}$ into E^3. Referring to an arbitrary origin point in E^3, the space curve is expressed as a vector function of one-parameter [45, 46]:

$$C : \boldsymbol{\alpha} = \boldsymbol{\alpha}(t). \tag{1.1.1}$$

In study of the kinematics, it is convenient to consider the real parameter t as time and the curve C as the trajectory of a moving point in Euclidean 3-space E^3. A curve is said to be regular if the components of $\boldsymbol{\alpha}$ differentiable and $\frac{d\boldsymbol{\alpha}}{dt} \neq \mathbf{0}$ over the interval I. The arc-length of a regular curve C is given by

$$s(t) = \int_{t_0}^{t} \left\| \frac{d\boldsymbol{\alpha}}{dt} \right\| dt. \tag{1.1.2}$$

Hence forth, assuming that the curve is regular, the arc-length s is chosen as a new parameter yielding $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. This representation has certain unique properties. For example, the derivatives of the position vector $\frac{d\boldsymbol{\alpha}}{ds}$ is a unit tangent vector $\mathbf{t}(s)= \frac{d\boldsymbol{\alpha}}{ds}$; $\|\mathbf{t}(s)\|= 1$ while the second derivative is a vector normal to the curve (termed the principal normal). Its magnitude is equal to the curvature κ

$$\frac{d\mathbf{t}(s)}{ds} = \kappa\mathbf{n}(s), \ \ \|\mathbf{n}(s)\|= 1. \qquad (1.1.3)$$

The unit tangent and unit principal normal vectors, together with the unit binormal vector defined by $\mathbf{t}(s)\times \mathbf{n}(s)= \mathbf{b}(s)$ satisfy the Frenet formulae:

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}, \qquad (1.1.4)$$

where $\tau(s)$ and $\kappa(s)$ are called the natural torsion and curvature functions of $\boldsymbol{\alpha}(s)$, respectively.

Bertrand offsets of curves

A pair of curves are said to be Bertrand offsets if there exists a one-to-one correspondence between their points such that both curves have a common principal normal at their

corresponding points. Such curves will be referred to as Bertrand offsets (mates). The problem of finding a curve whose principal normals are also the principal normals of another curve was, apparently, first proposed by Saint-Venant (see Weatherburn [45]) but solved by Bertrand. Here, a brief overview of the essential details is provided before a similar theory for ruled surfaces is developed.

Now consider a space curve $\boldsymbol{\alpha}$ and its Bertrand offset $\bar{\boldsymbol{\alpha}}$. If $\mathbf{n}$ denotes the principal normal of $\boldsymbol{\alpha}$, then $\bar{\boldsymbol{\alpha}}$ has principal normal $\bar{\mathbf{n}}$ which is the same as $\mathbf{n}$. A point on $\bar{\boldsymbol{\alpha}}$ corresponding to a point on $\boldsymbol{\alpha}$ is then given by

$$\bar{\boldsymbol{\alpha}}(s)= \boldsymbol{\alpha}(s)+\psi^*(s)\mathbf{n}(s), \qquad (1.1.5)$$

where $\psi^* = \psi^*(s)$ is the offset distance between the two corresponding points. Some of the essential properties of curves which are Bertrand offset can be described in terms of the following theorems:

Theorem (1.1.1): Two curves which are Bertrand offsets are constant offsets of one another.

Proof: Let $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ be a set of Bertrand offsets with s the arc-length on $\boldsymbol{\alpha}$. The tangent of the Bertrand offset $\bar{\boldsymbol{\alpha}}$ is parallel to $\bar{\mathbf{t}}$. This implies that $\bar{\mathbf{t}}$ is perpendicular to $\mathbf{n}$. Analytically, from

equation (1.1.5) and Frenet formulas

$$\bar{\mathbf{t}}\frac{d\bar{s}}{ds} = \mathbf{t} + \frac{d\psi^*}{ds}\mathbf{n} + \psi^*(\tau\mathbf{b} - \kappa\mathbf{t}),$$

for $\bar{\mathbf{t}}$, to be perpendicular to $\mathbf{n}$, $<\bar{\mathbf{t}},\mathbf{n}>= 0$ or $\frac{d\psi^*}{ds} = 0$. This gives $\psi^* = constant$. In other words, the offset distance between corresponding points of a curve and its Bertrand offset is a constant which proves the theorem.

It should be noted that the relationship between a curve and its Bertrand offset is a reciprocal one. In other words, if a curve $\bar{\boldsymbol{\alpha}}$ is a Bertrand offset of a curve $\boldsymbol{\alpha}$, then $\boldsymbol{\alpha}$ is also the Bertrand offset of $\bar{\boldsymbol{\alpha}}$.

Theorem (1.1.2): The tangents to corresponding points of a curve and its Bertrand offset make a constant angle.

Proof: It is first necessary to evaluate $\frac{d}{ds}<\bar{\mathbf{t}},\mathbf{t}>$.

$$\frac{d}{ds}<\bar{\mathbf{t}},\mathbf{t}>=<\frac{d\bar{\mathbf{t}}}{d\bar{s}},\mathbf{t}>\frac{d\bar{s}}{ds}+<\bar{\mathbf{t}},\frac{d\mathbf{t}}{ds}>$$

From Frenet formulas:

$$\frac{d}{ds}<\bar{\mathbf{t}},\mathbf{t}>= \bar{\kappa}<\bar{\mathbf{n}},\mathbf{t}>\frac{d\bar{s}}{ds} + \kappa<\bar{\mathbf{t}},\mathbf{n}>.$$

But since the principal normal $\mathbf{n}$ of a curve and that of its Bertrand offset $\bar{\mathbf{n}}$ coincide this last expression becomes:

$$\frac{d}{ds} < \bar{\mathbf{t}}, \mathbf{t} >= 0 \quad or \quad < \bar{\mathbf{t}}, \mathbf{t} >= \cos\psi = const.,$$

where ψ is the angle between $\bar{\mathbf{t}}$ and $\mathbf{t}$. This complete the proof.

In the view of the fact that for a curve and its Bertrand offset the principal normals coincide, it follows from the above theorem that the binormals of the two curves also make the same constant angle ψ at the corresponding points on the two curves. Therefore, we can write that:

$$\begin{bmatrix} \bar{\mathbf{t}} \\ \bar{\mathbf{n}} \\ \bar{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{1.1.6}$$

Theorem (1.1.3): If a curve $\boldsymbol{\alpha}$ has a Bertrand offset $\bar{\boldsymbol{\alpha}}$ then the following relationship can be written between its curvature and torsion:

$$\tau\psi^* \cos\psi - (1 - \kappa\psi^*) \sin\psi = 0, \tag{1.1.7}$$

where ψ is the constant angle between the tangents at corresponding points.

Proof: Let s be the arc-length along $\boldsymbol{\alpha}$. In the view of the fact that the offset distance ψ^* is a constant, it follows from equation (1.1.5) that

$$\frac{d\bar{\boldsymbol{\alpha}}}{ds} \coloneqq \bar{\mathbf{t}} \cdot \frac{d\bar{s}}{ds} = (1 - \psi^*\kappa)\mathbf{t} + \psi^*\tau\mathbf{b}. \tag{1.1.8}$$

Forming the scalar product of each side with $\bar{\mathbf{b}}$ and using the relationship (1.1.6):

$$\tau\psi^* \cos\psi - (1-\kappa\psi^*)\sin\psi = 0,$$

This complete the proof.

The relationship between the arc length of a curve and that of its Bertrand offset can be developed as follows: Let *s* be the arc length on the Bertrand offset of the curve $\boldsymbol{\alpha}$, then

$$\frac{d\bar{\boldsymbol{\alpha}}}{ds} = \frac{d\bar{\boldsymbol{\alpha}}}{d\bar{s}}\frac{d\bar{s}}{ds} = \bar{\mathbf{t}}\frac{d\bar{s}}{ds}$$

or, in view of equation (1.1.8):

$$\bar{\mathbf{t}}\frac{d\bar{s}}{ds} = (1-\psi^*\kappa)\mathbf{t} + \psi^*\tau\mathbf{b}$$

Substituting for $\bar{\mathbf{t}}$ from (1.1.6) and equating coefficients

$$\frac{d\bar{s}}{ds} = \frac{1-\psi^*\kappa}{\cos\,\psi} = \frac{\psi^*\tau}{\sin\,\psi}. \qquad (1.1.9)$$

In the case of plane curve $\tau = 0$ which in view of equation (1.1.7), implies that $\psi = 0$ and ψ^* can take any value. The first condition means that a plane curve and its Bertrand offset have parallel tangents at their corresponding points. For a space curve to have parallel tangents to its offset at the corresponding points, its offset ψ should be zero. Thus, in view of equation

(1.1.7), implies that $\psi^* = 0$ which means that the curve is coincident with its offset. A Bertrand offset of a space curve, therefore, does not have parallel tangents at the corresponding points.

(1.2) Differential Line Geometry

There is a tight connection between spatial kinematics and the line geometry in the three-dimensional Euclidean space E^3. Therefore we start with recalling the use of appropriate line coordinates: An oriented line L in Euclidean 3-space E^3 may be given by a point $\mathbf{a} \in L$ and a unit vector $\mathbf{x}$ on it, i.e. $\|\mathbf{x}\| = 1$. A parametric equation of the line is [8-10]:

$$L:\ \mathbf{b} = \mathbf{a} + \lambda \mathbf{x}, \quad \lambda \in \mathbb{R}, \qquad (1.2.1)$$

Then we define the moment of the vector $\mathbf{x}$ with respect to a fixed origin point in E^3 as:

$$\mathbf{x}^* = \mathbf{a} \times \mathbf{x} = \mathbf{b} \times \mathbf{x}, \qquad (1.2.2)$$

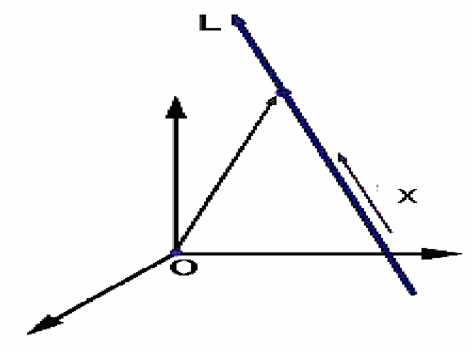

Fig. (1.2.1) : A line defined using Plücker coordinates

This means that the direction vector $\mathbf{x}$ of the line and its moment vector $\mathbf{x}^*$ are independent of the choice of the points of line. The two vectors $\mathbf{x}$ and $\mathbf{x}^*$ are not independent of one another; they satisfy the following relationships :

$$< \mathbf{x}, \mathbf{x} >= 1, \quad < \mathbf{x}, \mathbf{x}^* >= 0. \qquad (1.2.3)$$

The six components x_i, x_i^* (i =1, 2, 3) of $\mathbf{x}$ and $\mathbf{x}^*$ are Plukerian homogeneous line coordinates. Hence the two vectors $\mathbf{x}$, $\mathbf{x}^*$ determine the oriented line L, see Figure (1.2.1).

Conversely, any six-tuple x_i, x_i^*(i = 1, 2, 3) with

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0. \qquad (1.2.4)$$

represent a line in the three-dimensional Euclidean space E^3.

Theorem (1.2.1): There is one-to-one correspondence between the set of all oriented lines of ordered pairs of vectors $(\mathbf{x}, \mathbf{x}^*) \in E^3 \times E^3$ subject to the Plücker relationships (1.2.3).

According to relationships in (1.2.3), we immediately see that four of the six Plücker coordinates are independent. Then, the set of all oriented lines of E^3 constitutes a four dimensional manifold (space of lines).

(1.3) Dual numbers and vectors

The application of line geometry and dual number representation of line trajectories has been developed by Blaschke, W. [5], and Bottema, O. & Roth, B. [6]. A more recent descriptions of this representation can be found in the works [10, 36, 44, 47], the dual number is used to recast the point displacement relationship into relationships of lines.

A dual number is defined as an order pair of real numbers expressed formally as [27, 33]:

$$\mathrm{A} = \mathrm{a} + \varepsilon \mathrm{a}^*, \quad \varepsilon \neq 0, \tag{1.3.1}$$

where a is referred to as the real part and a^*as the dual part of A. The symbol ε is a multiplier which has the property:

$$\varepsilon \neq 0, \quad \varepsilon 1 = 1\varepsilon, \quad \varepsilon^2 = 0. \tag{1.3.2}$$

According to the definition pure dual numbers are zero divisors $(\varepsilon \mathrm{a}^*)(\varepsilon \mathrm{b}^*) = 0$. No number has an inverse in the algebra. But the other lows of the algebra of dual numbers are the same as the lows of the algebra of complex numbers $(\mathrm{a} + i\mathrm{b}, i^2 = -1)$. This means that the set of dual numbers

$$\mathrm{D} = \{\mathrm{A} = \mathrm{a} + \varepsilon \mathrm{a}^*; \quad \mathrm{a}, \mathrm{a}^* \in \mathbb{R},\ \varepsilon \neq 0,\ \varepsilon^2 = 0\}, \tag{1.3.3}$$

form a ring over the real numbers filed.

The formal operations of dual numbers are precisely those of ordinary algebra followed by setting $\varepsilon^2 = \varepsilon^3 = \cdots = 0$.

$$A = a + \varepsilon a^* = a(1 + \varepsilon Pa),$$
$$B = b + \varepsilon b^* = b(1 + \varepsilon Pb).$$
$$\text{Equality : } A = B \Leftrightarrow a = b,\ a^* = b^*,$$
$$\text{Addition : } A \pm B = (a \pm b) + \varepsilon(a^* \pm b^*),$$
$$\text{Multiplication : } AB = (a + \varepsilon a^*)(b + \varepsilon b^*),$$
$$= ab + \varepsilon(a^* b + ab^*). \qquad (1.3.4)$$

The division of dual numbers is defined as:

$$\frac{A}{B} = \frac{a+\varepsilon a^*}{b+\varepsilon b^*}.\frac{b-\varepsilon b^*}{b-\varepsilon b^*},$$
$$= \frac{a}{b} + \varepsilon\left(\frac{a^* b - ab^*}{b^2}\right),\quad b \neq 0. \qquad (1.3.5)$$

A dual number is called a purely dual if

$$A = \varepsilon a^*.$$

If a function f(t) has the derivative $f'(t)$, it's value for the dual variable $T = t + \varepsilon t^*$ is found by writing it's formal Taylor expansion with the property $\varepsilon^2 = 0$; thus:

$$f(T) = f(t + \varepsilon t^*) = f(t) + \varepsilon t^* f'(t). \qquad (1.3.6)$$

As a direct consequence of this, we have the expressions below. The trigonometric functions of Φ can be expanded with

the aid of (1.3.4):

$$\tan(\varphi + \varepsilon\varphi^*) = \tan\varphi + \varepsilon\varphi^*(1 + \tan^2\varphi),$$

$$\cot(\varphi + \varepsilon\varphi^*) = \cot\varphi - \varepsilon\varphi^*(1 + \cot^2\varphi),$$

and

$$\sin^{-1}(\varphi + \varepsilon\varphi^*) = \sin^{-1}\varphi + \varepsilon\frac{\varphi^*}{\sqrt{1-\varphi^2}},$$

$$\cos^{-1}(\varphi + \varepsilon\varphi^*) = \cos^{-1}\varphi - \varepsilon\frac{\varphi^*}{\sqrt{1-\varphi^2}},$$

$$\tan^{-1}(\varphi + \varepsilon\varphi^*) = \tan^{-1}\varphi + \varepsilon\frac{\varphi^*}{1+\varphi^2}.$$

Other functions may also be defined in this same manner. Furthermore, the dual exponential of e,

$$\mathrm{e}^{\mathrm{T}} = \mathrm{e}^{(\mathrm{t}+\varepsilon\mathrm{t}^*)} = \mathrm{e}^{\mathrm{t}}\mathrm{e}^{\varepsilon\mathrm{t}^*},$$

may be expressed, by (1.3.6), as:

$$\mathrm{e}^{\mathrm{T}} = \mathrm{e}^{\mathrm{t}} + \varepsilon\mathrm{t}^*\mathrm{e}^{\mathrm{t}} = \mathrm{e}^{\mathrm{t}}(1 + \varepsilon\mathrm{t}^*).$$

By equating the above two equations, we obtain the identity:

$$(1 + \varepsilon\mathrm{P}) = \mathrm{e}^{\varepsilon\mathrm{P}}, \qquad (1.3.7)$$

where P is real number. Using (1.3.7), any dual number A may be expressed in exponential form:

$$\mathrm{A} = \mathrm{a} + \varepsilon\mathrm{a}^* = \mathrm{a}(1 + \varepsilon\mathrm{P_a}) = \mathrm{a}\mathrm{e}^{\varepsilon\mathrm{P_a}};\ \left(\mathrm{P_a} = \frac{\mathrm{a}^*}{\mathrm{a}}\right), \qquad (1.3.8)$$

where P_a is called the parameter of A. When P_a is finite, A is aproper dual; P_a is zero, A reduces to a real number; and when P_a is infinite (a = 0); A is a pure dual.

It may also be shown that, for integer n,

$$A^n = a^n + \varepsilon n a^* a^{n-1} = a^n(1 + \varepsilon n P_a). \qquad (1.3.9)$$

$$A^{\frac{1}{n}} = a^{\frac{1}{n}} + \varepsilon \frac{a^*}{n a^{\frac{n-1}{n}}} = a^{\frac{1}{n}}\left(1 + \varepsilon \frac{1}{n} P_a\right). \qquad (1.3.10)$$

The absolute value | | of A is defined by:

$$|A| = |a| e^{\varepsilon P_a}. \qquad (1.3.11)$$

An example of dual number is the dual angle between two skew lines in space defined as $\Theta = \theta + \varepsilon\theta^*$: (see Figure (1.3.1) bellow).

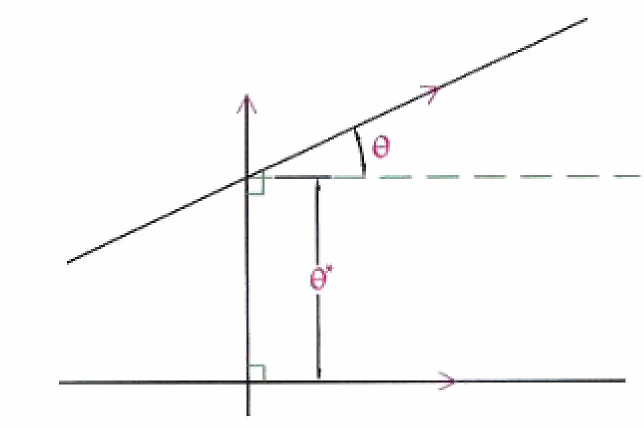

Fig. (1.3.1): Dual angle $\Theta = \theta + \varepsilon\theta^*$.

where θ is projected angle between the lines and θ^* is the minimal distance between the lines along their common perpendicular line.

(1.4) The E. Study's map

An ordered pair of vectors $(\mathbf{x}, \mathbf{x}^*) \in E^3$, which are generally not orthogonal to each other, can be composite into the dual vector

$$\mathbf{X} = \mathbf{x} + \varepsilon \mathbf{x}^*, \;\; \varepsilon \neq 0. \qquad (1.4.1)$$

where $\mathbf{x}$ and $\mathbf{x}^*$ are referred to the principal vector and the moment vector of the dual vector, respectively. The set of dual numbers can be extend to 3-dimensional vector space D^3 as:

$$D^3 = \{\mathbf{X} = \mathbf{x} + \varepsilon \mathbf{x}^*; \;\; \varepsilon \neq 0, \varepsilon^2 = 0; \;\; (\mathbf{x}, \mathbf{x}^*) \in E^3\},$$

This set is a module over the ring (D, +, .) and is called the dual space. Then for any $\mathbf{X}$ and $\mathbf{Y}$ in D^3, the scalar (inner) product is defined by:

$$< \mathbf{X}, \mathbf{Y} > = < \mathbf{x}, \mathbf{y} > + \varepsilon(< \mathbf{x}^*, \mathbf{y} > + < \mathbf{x}, \mathbf{y}^* >). \qquad (1.4.2)$$

Therefore, the norm of $\mathbf{X}$ is defined by:

$$\|\mathbf{X}\| = \|\mathbf{x}\| + \varepsilon \frac{<\mathbf{x}, \mathbf{x}^*>}{\|\mathbf{x}\|}, \;\; \|\mathbf{x}\| \neq \mathbf{0}. \qquad (1.4.3)$$

Hence, we may write the dual vector $\mathbf{X}$ as a dual multiplier of a unit dual vector in the form [7, 10, 36, 44, 47]:

$$\mathbf{X} = \|\mathbf{X}\|\mathbf{U}. \qquad (1.4.4)$$

Here the line $\mathbf{U} = \mathbf{u} + \varepsilon\mathbf{u}^*$ is referred to as the axis, $\|\mathbf{X}\|$ is dual magnitude and the ratio:

$$h = \frac{<\mathbf{x},\mathbf{x}^*>}{\|\mathbf{x}\|}, \qquad (1.4.5)$$

is called the pitch of $\mathbf{X}$. The pitch can be used to characterize the dual vector. For example, when h is finite, $\mathbf{X}$ is called a proper dual; when $h = 0$, and $\|\mathbf{x}\| = 1$ then $\mathbf{X}$ is a dual unit vector or oriented line. Consequently, each oriented line is $\mathrm{L} = (\mathbf{x}, \mathbf{x}^*) \in \mathrm{E}^3$ represented by dual unit vector in D^3-space if and only if :

$$< \mathbf{X}, \mathbf{X} >=< \mathbf{x}, \mathbf{x} > +2\varepsilon < \mathbf{x}, \mathbf{x}^* >= 1. \qquad (1.4.6)$$

Hence, it follows that the relationships (1.2.3) and equation (1.4.6) are corresponding. Now, we introduce the dual unit sphere in D^3-space as follow:

$$\mathrm{K} = \{\mathbf{X} = \mathbf{x} + \varepsilon\mathbf{x}^*;\ \ \mathbf{x}, \mathbf{x}^* \in \mathrm{E}^3, \|\mathbf{X}\| = 1\}.$$

Via this we have the following theorem [10, 21]:

Theorem (1.4.1) (E. Study): The set of oriented lines in E^3 is in one-to-one correspondence with the set of points on dual unit

sphere in the D^3-space.

This dualized form of line representation along with the E. Study's map leads to a new interpretation of the scalar and vectorial products of two lines. For two directed lines **X** and **Y** the dual angle combines the angle φ and the minimal distance φ^*. This gives rise to geometric interpretations of the following products of the dual unit vectors:

Theorem (1.4.2): $<\mathbf{X},\mathbf{Y}>$ is the cosine of the dual angle $\Phi=\varphi+\varepsilon\varphi^*$ of the two lines **X** and **Y** $(<\mathbf{X},\mathbf{Y}>=\cos\Phi)$.

Proof*:* According to equation (1.4.2) we may express this theorem as follows:

$$\begin{aligned}<\mathbf{X},\mathbf{Y}>&=\cos\Phi\\ &=<\mathbf{x},\mathbf{y}>+\varepsilon[<\mathbf{x},\mathbf{y}^*>+<\mathbf{x}^*,\mathbf{y}>],\end{aligned}\qquad(1.4.7)$$

i.e. from (1.3.6) we must show that:

$$<\mathbf{X},\mathbf{Y}>=\cos\Phi=\cos\varphi-\varepsilon\varphi^*\sin\varphi.\qquad(1.4.8)$$

Then the equations (1.4.7), and (1.4.8) give us:

$$<\mathbf{x},\mathbf{y}>=\cos\varphi,\;\; <\mathbf{x},\mathbf{y}^*>+<\mathbf{x}^*,\mathbf{y}>=-\varepsilon\varphi^*\sin\varphi.\;(1.4.9)$$

Since φ the angle between the two lines the first equality of (1.4.9) is obvious. In order to show the second equality of (1.4.9) we choose points **a** on **X** and **b** on **Y** such that they are

lie on the common perpendicular between the two lines $\mathbf{X}$ and $\mathbf{Y}$, i.e. $\mathbf{b}-\mathbf{a}=\varphi^*\mathbf{n}$, see Figure (1.4.1). Then:

$$\begin{aligned} <\mathbf{x},\mathbf{y}^*> + <\mathbf{x}^*,\mathbf{y}> &= \det[\mathbf{x},\mathbf{b},\mathbf{y}] + \det[\mathbf{a},\mathbf{x},\mathbf{y}], \\ &= \det[\mathbf{x},\mathbf{y},\mathbf{a}-\mathbf{b}], \\ &= -\varphi^* \sin\varphi. \end{aligned}$$

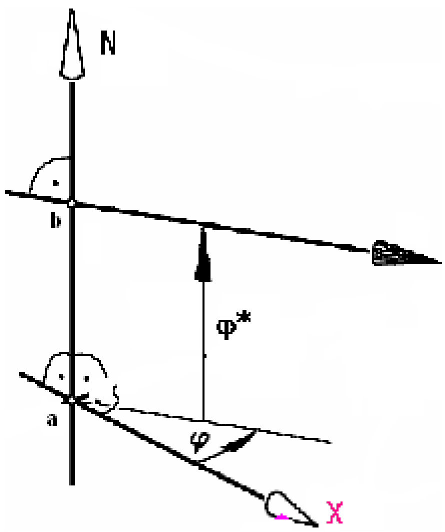

Fig. (1.4.1): $<\mathbf{X},\mathbf{Y}> = \cos\Phi$.

Hence, we have:

$$<\mathbf{X},\mathbf{Y}> = \cos\varphi - \varepsilon\varphi^*\sin\varphi = \cos\Phi, \qquad (1.4.10)$$

and the following special cases are very important:

1- If $<\mathbf{X},\mathbf{Y}> = 0$, then $\varphi=\frac{\pi}{2}$ and $\varphi^*=0$; this means that the two lines $\mathbf{X}$ and $\mathbf{Y}$ meet at right angle,

2- If $<\mathbf{X},\mathbf{Y}>=$pure dual, then $\varphi=\frac{\pi}{2}$ and $\varphi^*\neq 0$; the lines $\mathbf{X}$ and $\mathbf{Y}$ are orthogonal skew lines,

3- If $<\mathbf{X},\mathbf{Y}>=$pure real, then $\varphi\neq\frac{\pi}{2}$ and $\varphi^*=0$; the lines $\mathbf{X}$ and $\mathbf{Y}$ are intersect,

4- If $<\mathbf{X},\mathbf{Y}>=1$, then $\varphi=0$ and $\varphi^*=0$; the lines $\mathbf{X}$ and $\mathbf{Y}$ are coincident (their directions are the same or opposite).

Then, for two lines $\mathbf{X}$ and $\mathbf{Y}$, we find from the Fig. (1.4.1) that their vectorial product is:

$$\begin{aligned}\mathbf{X}\times\mathbf{Y}&=\mathbf{x}\times\mathbf{y}+\varepsilon(\mathbf{x}\times\mathbf{y}^*+\mathbf{x}^*\times\mathbf{y})\\&=\mathbf{n}\sin\varphi+\varepsilon[\mathbf{x}\times(\mathbf{b}\times\mathbf{y})+(\mathbf{a}\times\mathbf{x})\times\mathbf{y}],\end{aligned}$$

where $\mathbf{b}-\mathbf{a}=\varphi^*\mathbf{n}$.

Observing that $\mathbf{b}=\mathbf{a}+\varphi^*\mathbf{n}$, we get:

$$\begin{aligned}\mathbf{X}\times\mathbf{Y}=\mathbf{n}\sin\varphi+\varepsilon[\mathbf{x}\times(\mathbf{a}\times\mathbf{y})+\mathbf{x}\times(\varphi^*\mathbf{n}\times\mathbf{y})\\+(\mathbf{a}\times\mathbf{x})\times\mathbf{y}],\end{aligned}$$

Since

$$\mathbf{x}\times(\mathbf{a}\times\mathbf{y})+\mathbf{a}\times(\mathbf{y}\times\mathbf{x})+\mathbf{y}\times(\mathbf{x}\times\mathbf{a})=0,$$

we find:

$$\begin{aligned}\mathbf{X}\times\mathbf{Y}&=\mathbf{n}\sin\varphi+\varepsilon[\mathbf{a}\times(\mathbf{x}\times\mathbf{y})+\varphi^*<x,\mathbf{y}>\mathbf{n}],\\&=\mathbf{n}\sin\varphi+\varepsilon[\mathbf{n}^*\sin\varphi+\varphi^*\mathbf{n}\cos\varphi],\\&=(\mathbf{n}+\varepsilon\mathbf{n}^*)(\sin\varphi+\varepsilon\varphi^*\cos\varphi),\\&=\mathbf{N}\sin\Phi.\end{aligned}\tag{1.4.11}$$

Theorem (1.4.3): Let two oriented lines $\mathbf{X}$ and $\mathbf{Y}$ meet at right angle, then

$$\mathbf{Z} = \cos\Phi\,\mathbf{X} + \sin\Phi\mathbf{Y}, \qquad (1.4.12)$$

defines an oriented line which is the image of $\mathbf{X}$ under a helical motion about the axis $\mathbf{X} \times \mathbf{Y}$ with the dual angle Φ (see Figure (1.4.2)).

Proof: From the perpendicular intersection of $\mathbf{X}$ and $\mathbf{Y}$ we conclude $< \mathbf{X}, \mathbf{Y} >= 0$ and $\mathbf{X} \times \mathbf{Y} = \mathbf{N}$. These equations imply

$$< \mathbf{Z}, \mathbf{Z} >= \cos^2\Phi < \mathbf{X}, \mathbf{X} > + \sin^2\Phi < \mathbf{Y}, \mathbf{Y} >= 1,$$

and from equation (1.4.11) we conclude

$$\mathbf{X} \times \mathbf{Z} = \mathbf{X} \times \mathbf{Y} \sin\Phi = \mathbf{N} \sin\Phi, \qquad (1.4.12)$$

as stated.

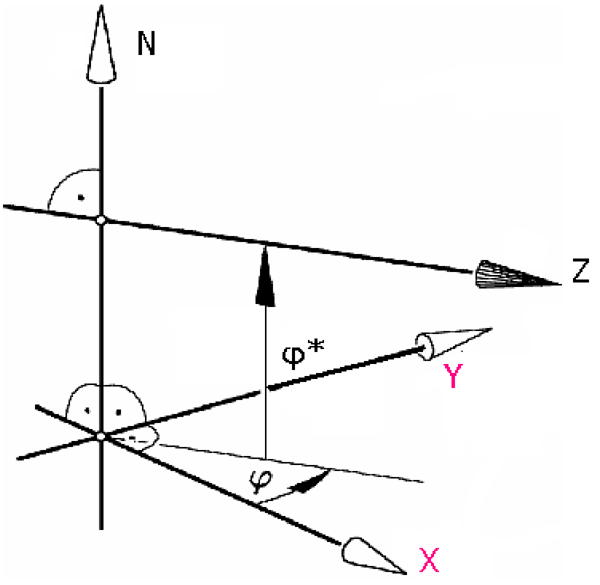

Fig. (1.4.2): $\mathbf{Z} = \cos\Phi\,\mathbf{X} + \sin\Phi\mathbf{Y}$.

CHAPTER 2

Differential geometry of ruled surface

This chapter study the curvature theory of ruled surfaces in two different ways. The scalar formulation proceeds by defining the classical Blaschke's frame associated with the trajectory ruled surface. While most of the theory is well known, the purpose is to present the theory in a manner that is appropriate for our study. For example, the positional variation and the angular variation characterize the local properties of the ruled surface. The other formulation uses dual vector algebra to a transform the differential geometry of ruled surface into that of dual spherical curve.

(2.1) The Classical Blaschke frame

A ruled surface is the surface which is generated by moving a straight line continuously in the space and is one of the most important topics of differential geometry. The straight lines are called the rulings or generators of the surface: Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ be a regular curve and $\mathbf{e} = \mathbf{e}(u)$ be a unit direction vector of an oriented line L in Euclidean 3-space E^3. Then we have the following parameterization for a ruled surface *M*:

$$M: \ \mathbf{L}(u,\mu) = \boldsymbol{\alpha}(u) + \mu\mathbf{e}(u), \qquad (2.1.1)$$

where μ is an arbitrary real-valued parameter, and u is taken as the motion parameter. The parametric μ-curve of this surface is a straight line of the surface which is called ruling. For $\mu = 0$, the parametric μ-curve of this surface is $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ which is called base curve or the directrix curve of the surface. We can, therefore, select any space curve lying on the ruled surface and cutting all the generators as the directrix. In particular, if $\mathbf{e}$ is constant, the ruled surface is said to be cylindrical, and non-cylindrical otherwise.

The unit surface normal $\mathbf{N}$ to a ruled surface represented by equation (2.1.1), and denoting derivatives with respect to du by dash, is then

$$\mathbf{N}(u,\mu) = \frac{(\boldsymbol{\alpha}' + \mu\mathbf{e}')\times\mathbf{e}}{\sqrt{\|(\boldsymbol{\alpha}' + \mu\mathbf{e}')\|^2 - [<\boldsymbol{\alpha}'(u), \mathbf{e}(u)>]^2}}. \qquad (2.1.2)$$

The unit normal along a general generator $\mathbf{L}(u_0,\mu)$ of the ruled surface approaches a limiting direction as μ infinitely decreases. This direction is called the central tangent (asymptotic) direction and is defined as:

$$\mathbf{N}(u,\mu)|_{\substack{u=u_0 \\ u\to-\infty}} = \frac{-\mathbf{e}'\times\mathbf{e}}{\|\mathbf{e}'\|}|_{u=u_0}. \qquad (2.1.3)$$

As μ increases to $+\infty$, the unit normal rotates through 180° about **L** and ultimately takes the direction $-\mathbf{N}(u,\mu)|_{\substack{u=u_0 \\ u\to-\infty}}$. The point at which **N** has rotated only 90° and is perpendicular to $\mathbf{N}(u,\mu)|_{\substack{u=u_0 \\ u\to-\infty}}$ is called the striction point (or central point) on **L**. The direction of **N** at this point is called the central normal of the ruled surface. The set of the central points constitute a curve lying on the ruled surface and is called striction curve.

The differential geometry of surfaces uses the angular variation of a natural reference frame on the surface measured relative to itself to characterize its local properties. The equations which result are the structure equations of the surface and are a generalization of the Frenet equations of a regular space curve. General surfaces do not have the characteristic striction curve that a ruled surface has. Thus, the Blaschke frame on a ruled surface can then be defined by the dexterous triplet of vectors: The generator vector $\mathbf{e}_1 = \mathbf{e}(u)$, the central normal vector is defined as $\mathbf{e}_2 = \frac{\mathbf{e}'}{\|\mathbf{e}'\|}$ and the central tangent vector $\mathbf{e}_3 = \mathbf{e} \times \frac{\mathbf{e}'}{\|\mathbf{e}'\|}$. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called the Blaschke moving frame of the ruled surface as introduced by Blaschke, W. [5], and developed by Sannia, G. [8, 21, 30].

The striction curve $\mathbf{C}(u)$ is unique and can be written in terms of the directrix $\boldsymbol{\alpha}(u)$ as

$$\mathbf{C}(u) = \boldsymbol{\alpha}(u) + \xi(u)\mathbf{e}_1(u), \qquad (2.1.4)$$

where $\xi(u)$ is a smooth function. To determine $\xi(u)$, we use the defintion of the strciction curve given by:

$$< \mathbf{C}'(u), \mathbf{e}_1' >= 0. \qquad (2.1.5)$$

This last equation implies that the tangent to the striction curve $\mathbf{C}'(u)$ is perpendicular to the central normal $\mathbf{e}_2$ of the ruled surface. Analytically this condition can be written as:

$$< \mathbf{C}'(u), \mathbf{e}_2 >= 0. \qquad (2.1.6)$$

Differentiating equation (2.1.4), the first-order positional variation of the striction curve is

$$\mathbf{t}(s) := \mathbf{C}'(u) = \boldsymbol{\alpha}'(u) + \xi'(u)\mathbf{e}_1(u) + \xi(u)\mathbf{e}_1'(u). \qquad (2.1.7)$$

Substituting equation (2.1.7) into equation (2.1.5) gives

$$0 =< \boldsymbol{\alpha}'(u), \mathbf{e}_1'(u) > +\xi'(u) < \mathbf{e}_1(u), \mathbf{e}_1'(u) > + \\ \xi(u) < \mathbf{e}_1'(u), \mathbf{e}_1'(u) >. \qquad (2.1.8)$$

Since $< \mathbf{e}_1(u), \mathbf{e}_1'(u) >= 0$, we have

$$\xi(u) = -\frac{<\boldsymbol{\alpha}'(u),\mathbf{e}_1'(u)>}{\|\mathbf{e}_1'(u)\|^2}. \qquad (2.1.9)$$

Moreover, we can choose the parameter u to be the arc-length along the striction curve $\mathbf{C}(u)$ and since the ruled surface is not developable, we may take $\mathbf{C}(u)$ to be the striction curve of the ruled surface. The ruled surface can then be written as [26]:

$$M:\ \mathbf{L}(u,\mu) = \mathbf{C}(u) + \mu\mathbf{e}_1(u), \quad \mu \in \mathbb{R}. \qquad (2.1.10)$$

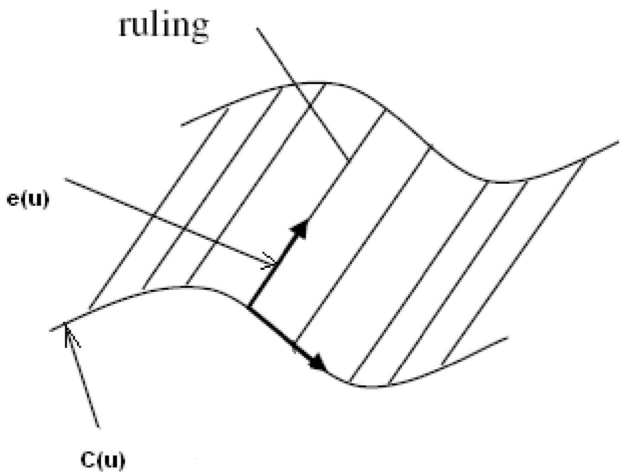

Figure (2.1.1): Parametric representation of ruled Surface

Let H_m be moving space generated by the Blaschke's frame relative to M, that is $\{\mathbf{C};\ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and H_f be a fixed space represented by set of orthonormal frame $\{0_f;\ \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ in Euclidean 3-space E^3. In this case, the structural equation of

one-parameter spatial motion H_m/H_f (read of H_m moving with respect to H_f) is defined by [1-3]:

$$H_m/H_f : \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} 0 & p & 0 \\ -p & 0 & q \\ 0 & -q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}, \qquad (2.1.11)$$

where

$$p = \|\mathbf{e}'\|, \quad q = \frac{\det[\mathbf{e}, \mathbf{e}', \mathbf{e}'']}{\|\mathbf{e}'\|^3}, \qquad (2.1.12)$$

are called the Blaschke invariants or curvature functions of the ruled surface. Also, the first-order positional variation of the striction curve may be expressed in the Blaschke frame as

$$\mathbf{t}(u) = <\mathbf{t}, \mathbf{e}_1> \mathbf{e}_1 + <\mathbf{t}, \mathbf{e}_2> \mathbf{e}_2 + <\mathbf{t}, \mathbf{e}_3> \mathbf{e}_3. \qquad (2.1.13)$$

To determine the coefficient in the second bracket-term, using equation (2.1.6), gives

$$\mathbf{t}(u) = <\mathbf{t}, \mathbf{e}_1> \mathbf{e}_1 + <\mathbf{t}, \mathbf{e}_3> \mathbf{e}_3. \qquad (2.1.14)$$

Normalization. An arbitrary real valued parameter $u \in \mathbb{R}$, was used in equation (2.1.1) as the independent variable for the ruled surface *M*. In order to simplify and standardize the formulation, a new normalized parameter will replace this parameter. There are two ways to normalize the parameter. First, the normalized parameter may be choose the parameter u

as the arc-length of the striction curve, an additional invariant is the striction angle $\sigma = \cos^{-1} < \mathbf{t}, \mathbf{e}_1 >$ measuring the deviation of the generator **e** from the striction curve $\mathbf{C}(u)$; it is determined by

$$\mathbf{t}(u) = \cos\sigma \mathbf{e}_1 + \sin\sigma\, \mathbf{e}_3, \qquad (2.1.15)$$

and may be restricted to the interval $-\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2}$. Second, it is convenient to use the arc length of the spherical indicatrix $\mathbf{e}_1(u)$ as this normalized parameter. The arc length parameter ϕ is defined by the equation

$$\phi(u) = \int_0^u p du. \qquad (2.1.16)$$

In what follows we will use both of these possibilities according to which of the two will be more advantageous in the given case. If $p \neq 0$ then equation (2.1.16) can be inverted to yield $u(\phi)$ allowing the defintion of $\mathbf{e}_1(\phi(u)) = \mathbf{e}_1(\phi). \mathbf{e}_1(\phi)$ has unit speed, that is its tangent vector is of unit length. Therefore, from equation (2.1.11), the Blaschke formulae may be expressed as:

$$H_m/H_f\ : \frac{d}{d\phi}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & \gamma\\ 0 & -\gamma & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}. \qquad (2.1.17)$$

The function $\gamma = \frac{q}{p}$ is called the geodesic curvature of the spherical indicatrix $\mathbf{e}_1(u)$. Also, using (2.1.15) and the chain rule for differentiation we compute

$$\frac{d\mathbf{C}}{d\phi} = \Gamma\mathbf{e}_1 + \delta\mathbf{e}_3, \tag{2.1.18}$$

where

$$\delta = \frac{\sin\,\sigma}{p}, \quad \Gamma = \frac{\cos\,\sigma}{p}.$$

Thus the three functions $\gamma = \gamma(\phi)$, $\delta = \delta(\phi)$ and $\Gamma = \Gamma(\phi)$ are referred as the curvature functions of a ruled surface and completely determine the ruled surface *M*. These results are analogous to results found in Hoschek (1971) and Kruppa (1957) which are presented using so-called Kruppa invariants.

The results derived thus far provide a kinematic interpretation of the shape of the surface *M* in terms of the motion of a line, directed along $\mathbf{e}_1(u)$ and passing through the striction curve $\mathbf{C}(u)$. At a particular instant $u = u_0$ the velocity of the line is given by its instantaneous rotation about the central tangent vector $\mathbf{e}_3$ together with its instantaneous translation along $\mathbf{e}_3$. The ratio of these two quantities is the distribution δ.

(2.2) Developable ruled surface

A ruled surface, as defined by equation (2.1.1) is said to be a developable if and only if at all points

$$\det[\boldsymbol{\alpha}', \mathbf{e}_1, \mathbf{e}_1'] = 0. \qquad (2.2.1)$$

From equation (2.2.1), we note that there are two possible cases: The first case is when

$$\mathbf{e}_1 \times \mathbf{e}_1' = \mathbf{0}. \qquad (2.2.2)$$

Since the ruling has unit magnitude, the dot (scalar) product is

$$< \mathbf{e}_1, \mathbf{e}_1 >= 1.$$

Differentiating above equation gives

$$< \mathbf{e}_1, \mathbf{e}_1' >= 0,$$

which shows either $\mathbf{e}_1$ and $\mathbf{e}_1'$ are always mutually perpendicular or $\mathbf{e}_1'$ is zero. From equation (2.2.2) we conclude that $\mathbf{e}_1'$ is zero and therefore, is a constant vector. In this case, the ruled surface is referred to as a cylindrical surface.

The second case is when

$$\mathbf{e}_1 \times \mathbf{e}_1' \neq \mathbf{0}. \qquad (2.2.3)$$

This implies that the ruled surface is a non-cylindrical ruled

surface. From equation (2.1.4), the first-order derivative of the directrix is

$$\boldsymbol{\alpha}'(u) = \mathbf{C}'(u) - \xi'(u)\mathbf{e}_1(u) - \xi(u)\mathbf{e}_1'(u). \qquad (2.2.4)$$

where $\mathbf{C}'(u)$ is the first-order positional variation of the striction curve. Substituting equation (2.2.4) into equation (2.2.1), and using the fact that $\mathbf{e}_1'$ is perpendicular to $(\mathbf{e}_1 \times \mathbf{e}_1')$, gives

$$\det[\mathbf{C}', \mathbf{e}_1, \mathbf{e}_1'] = 0. \qquad (2.2.5)$$

There are two possible cases which satisfy equation (2.2.5), as presented in the following paragraphs. The first-order positional variation of the striction curve is

$$\mathbf{C}'(u) = \mathbf{0}. \qquad (2.2.6)$$

Equation (2.2.6) implies that the striction curve degenerates to a point, and the ruled surface becomes a cone. In terms of geometry, the striction point of a cone is commonly referred to as the vertex. The second case is when the firs-order positional variation of the stricion curve is

$$\mathbf{C}'(u) \neq \mathbf{0}. \qquad (2.2.7)$$

From equation (2.2.5), $\mathbf{C}'(u)$ is perpendicular to $(\mathbf{e}_1 \times \mathbf{e}_1')$, and

therefore, $\mathbf{C}'(u)$ is in the plane generated by $\mathbf{e}_1$ and $\mathbf{e}_1'$. From the definition of striction curve, see equation (2.1.5), $\mathbf{C}'$ and $\mathbf{e}_1'$ are perpendicular to each other. Therefore, we may conclude that the ruling is parallel to the first-order positional variation of the striction curve, which is also the tangent vector of the striction curve. In this case the striction curve $\mathbf{C}(u)$ is called the edge of regression, and M is the surface of the tangents to a space curve-tangential tourse. Namely, $\mathbf{e}_1$ is tangent to the edge of regression of the surface and $\mathbf{e}_2$ and $\mathbf{e}_3$ will be the unit principal normal and binormal vectors of the striction curve, respectively. Then p and q are the usual curvature κ and the torsion τ functions of the striction curve, respectively. Therefore, we have:

$$\gamma = \frac{\tau}{\kappa}, \quad \delta = 0, \quad \Gamma = \frac{1}{\tau}. \tag{2.2.8}$$

(2.3) The dual Blaschke frame

In this subsection we present the differential geometry of ruled surface in terms of three dimensional dual vector calculus. The result is a set of dual functions which characterize the ruled surface, (see [11-15]). Dual vector calculus allows to rewrite equation (2.1.10) by the dual vector function as:

$$M: \ \mathbf{E}_1(u) = \mathbf{e}_1(u) + \varepsilon \mathbf{C}(u) \times \mathbf{e}_1(u), \tag{2.3.1}$$

since the spherical image $\mathbf{e}_1$ is a unit vector, the dual vector $\mathbf{E}_1$ also has unit length as is seen from the computation

$$
\begin{aligned}
<\mathbf{E}_1,\mathbf{E}_1> &=< \mathbf{e}_1+\varepsilon\,\mathbf{C}\times\mathbf{e}_1,\mathbf{e}_1+\varepsilon\,\mathbf{C}\times\mathbf{e}_1> \\
&=<\mathbf{e}_1,\mathbf{e}_1>+2\varepsilon<\mathbf{e}_1,\mathbf{C}\times\mathbf{e}_1>+\varepsilon^2<\mathbf{C}\times\mathbf{e}_1,\mathbf{C}\times\mathbf{e}_1> \\
&=<\mathbf{e}_1,\mathbf{e}_1>=1.
\end{aligned}
$$

The differentiable curve $\mathbf{E}_1=\mathbf{E}_1(u)$ represents a differentiable family of straight lines of Euclidean 3-space E^3. The lines $\mathbf{E}_1(u)$ are the generators of a surface. Hence, ruled surfaces and dual curves are synonymous in this work.

An orthonormal frame for $\mathbf{E}_1=\mathbf{E}_1(u)$ is found immediately:

$$
\left.
\begin{aligned}
\mathbf{E}_1 &= \mathbf{E}(u), \\
\mathbf{E}_2(u) &= \frac{\mathbf{E}_1'}{\|\mathbf{E}_1'\|}, \\
\mathbf{E}_3(u) &= \mathbf{E}_1\times\mathbf{E}_2.
\end{aligned}
\right\}
\tag{2.3.2}
$$

Hence, the dual frame $\{F\}$ (called the fixed dual frame), which is composed of the three mutually orthogonal oriented lines $\{\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3\}$, is rigidly attached to the fixed space H_f. Also, the dual frame $\{E\}$, which is composed of the three mutually orthogonal oriented lines $\{\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3\}$, is rigidly attached to the moving space H_m, in the 3-dimensional Euclidean space E^3 such that:

$$\mathbf{E}_i = \mathbf{e}_i + \varepsilon \mathbf{e}_i^*, \text{ and } \mathbf{F}_i = \mathbf{f}_i + \varepsilon \mathbf{f}_i^*, \ (i = 1, 2, 3) \tag{2.3.3}$$

where

$$\mathbf{e}_i^* = \mathbf{C} \times \mathbf{e}_i, \text{ and } \mathbf{f}_i^* = \mathbf{0}\mathbf{0}_f \times \mathbf{f}_i, \tag{2.3.4}$$

in which $\mathbf{0}$ is a fixed point as origin of E^3. Veldkamp [18] assumed that both of these frames are attached to separate dual unit spheres K_f and K_m with the same center $\mathbf{O}$ in the dual 3-space D^3. Then we say that $\{\mathbf{O};\ \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ moves with respect to $\{\mathbf{O};\ \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$. We may interpret this as follows: the dual unit sphere K_m rigidly connected with $\{\mathbf{O};\ \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ moves over the dual unit sphere K_f rigidly connected with $\{\mathbf{O}; \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$. This motion is called one-parameter dual spherical motion and will denoted by K_m / K_f. Therefore the dual Blaschke formulae, corresponding to the equations (2.1.11) and (2.1.12), can be written as [1-3]:

$$K_m / K_f :\ \begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix} = \begin{bmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \tag{2.3.5}$$

where

$$\left.\begin{aligned} P &= p + \varepsilon p^* = p + \varepsilon \sin\sigma = \|\mathbf{E}'\|, \\ Q &= q + \varepsilon q^* = q + \varepsilon \cos\sigma = \frac{\det(\mathbf{E}, \mathbf{E}', \mathbf{E}'')}{\|\mathbf{E}'\|^3}, \end{aligned}\right\} \tag{2.3.6}$$

are called the dual curvature functions of the ruled surface $\mathbf{E}(u)$ and their derivatives define the shape of the ruled surface $\mathbf{E}(u)$ in vicinity of a line $\mathbf{E}_1 = \mathbf{e}_1 + \varepsilon \mathbf{e}_1^*$. The dual angle between two neighboring generators $\mathbf{E}(u)$ and $\mathbf{E}(u + du)$ is shown in Fig. (2.3.1); this dual angle is called the differential dual arc element of the ruled surface and is denoted:

$$d\Phi \coloneqq d\phi + \varepsilon d\phi^* = (1 + \varepsilon\delta)d\phi = \|\mathbf{E}'\| du, \qquad (2.3.7)$$

where $d\phi$ is the projected angle or central angle between $\mathbf{E}(u)$ and $\mathbf{E}(u + du)$, $d\phi^*$ is the shortest distance between them. The dual part of the dual arc element δ is the distribution parameter of the ruled surface, see equation (2.1.18).

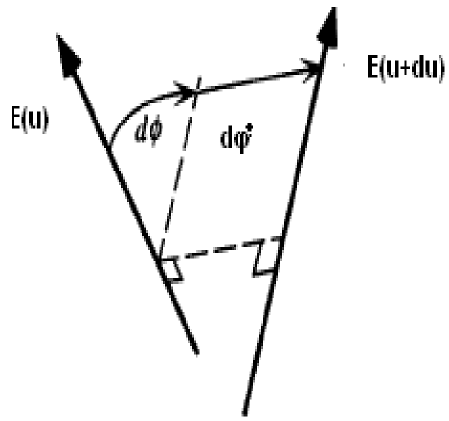

Fig. (2.3.1): Differential dual arc element.

Thus as in the case of spherical curves the dual arc length parameter normalizes the parameterization of $\mathbf{E}(u)$ such that its dual tangent $\mathbf{E}_2$ has unit length. Using (2.3.6) we derive the dual form of the Balschke frame in exactly the same way as the equation (2.1.17):

$$\frac{d}{d\Phi}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & \Sigma\\ 0 & -\Sigma & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}, \tag{2.3.8}$$

where

$$\Sigma = \gamma + \varepsilon\gamma^* = \frac{\det[\mathbf{E},\mathbf{E}',\mathbf{E}'']}{||\mathbf{E}'||^3}, \tag{2.3.9}$$

is called the dual spherical radius of curvature of the ruled surface [..]. Veldkamp also showed that the dual geodesic curvature can be written as:

$$\Sigma = \gamma + \varepsilon\gamma^* = \gamma + \varepsilon(\Gamma - \gamma\delta). \tag{2.3.10}$$

It is easy to see the dual vector calculus is a convenient tool for the analysis of ruled surfaces in the form of dual spherical curves. The derivations consistently yield formulas which are identical to those obtained in the differential geometry of curves on a unit sphere. In a way this tool is too efficient, it suppresses important geometric concepts such as the striction curve and the distribution parameter as well as the dual angle of pitch and the

other curvature functions. It seems clear that no matter how the ruled surface is represented the curvature functions p, q, and σ, or the equivalent set $\{p, p^*, q, q^*\}$ contain the fundamental geometric information describing the shape of the surface. In this sense, by using the dual vector calculus as [11-15, 24-29], we derived the scalar and dual formulations of the curvature theory of line trajectories and expose the fundamental curvature functions that characterize the shape of a ruled surface in the Euclidean 3-space.

(2.4) An Example and Remarks

For the one-parameter dual spherical motion K_m/K_f, let

$$C = \{\mathbf{X} \mid <\mathbf{X}, \ \mathbf{F}_1 >= const., \ \mathbf{X} \in K_m\},$$

be a dual curve on K_f. Thus, the dual unit vector $\mathbf{X}$ can be expressed as

$$\mathbf{X} = \cos\Theta \, \mathbf{F}_1 + \sin\Theta \cos\Phi \, \mathbf{F}_2 + \sin\Theta \sin\Phi \, \mathbf{F}_3, \qquad (2.4.1)$$

where

$$\Theta = \theta + \varepsilon\theta^*, \quad \Phi = \varphi + \varepsilon\varphi^*.$$

This means that

$$\theta = c_1(\text{real } const.), \quad \theta^* = c_2(\text{real } const.).$$

Thus equation (2.4.1) has only two real parameters φ and φ^*. So, if we choose $\varphi^* = h\varphi$, h denoting to the pitch of the motion H_m/H_f, and φ as the motion parameter, then equation (2.4.1) represents a ruled surface in H_f-space. Thus, Blaschke's frame of the ruled surface $\mathbf{X} = \mathbf{X}(\varphi)$ is found as

$$\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix} = \begin{bmatrix}\cos\Theta & \sin\Theta\cos\Phi & \sin\Theta\sin\Phi\\ 0 & -\sin\Phi & \cos\Phi\\ \sin\Theta & -\cos\Theta\cos\Phi & -\cos\Theta\sin\Phi\end{bmatrix}\begin{bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\\ \mathbf{F}_3\end{bmatrix}, \quad (2.4.2)$$

If we differentiate these expressions, we get:

$$\frac{d}{d\varphi}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix} = \begin{bmatrix}0 & P & 0\\ P & 0 & Q\\ 0 & -Q & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}, \quad (2.4.3)$$

where

$$P = (1+\varepsilon h)\sin\Theta,\ Q = (1+\varepsilon h)\cos\Theta. \quad (2.4.4)$$

Therefore, we have:

$$\Sigma = \frac{\cos\ \Theta}{\sin\ \Theta},\quad \gamma = \frac{\cos\ \theta}{\sin\ \theta}, \quad (2.4.5)$$

and

$$\delta = h + \theta^*\cot\theta,\quad \Gamma = h\cot\theta - \theta^*, \quad (2.4.6)$$

Now we may derivative the equation of the ruled surface $\mathbf{X} = \mathbf{X}(\varphi)$ in terms of the Plücker coordinates. Let $\mathbf{L}$ denote a point on this surface. We can write:

$$\mathbf{L}(\varphi,\mu)=\mathbf{X}(\varphi)\times\mathbf{X}^*(\varphi)+\mu\mathbf{X}(\varphi),\quad \mu\in\mathbb{R}. \qquad (2.4.7)$$

If (l_1,l_2,l_3) are the coordinates of **L**, then equations (2.4.1) and (2.4.7) yield

$$\left.\begin{aligned} l_1 &= \varphi^*\sin^2\theta+\mu\cos\vartheta,\\ l_2 &= -\theta^*\sin\varphi-(\varphi^*\cos\theta-\mu)\sin\theta\cos\varphi,\\ l_3 &= \theta^*\cos\varphi-(\varphi^*\cos\theta-\mu)\sin\theta\sin\varphi. \end{aligned}\right\} \qquad (2.4.8)$$

In the case of $0<\theta<\frac{\pi}{2}$ and $\theta^*\neq 0$ equations (2.4.8) gives us

$$\frac{l_2^2}{c_2^2}+\frac{l_3^2}{c_2^2}-\frac{L_1^2}{\kappa^2}=1,\ \ \kappa=c_2\cot c_1,\ \ L_1=l_1-\varphi^*. \qquad (2.4.9)$$

Equation (2.4.9) represents a hyperboloid of one-sheet; the generators of which are located as:

a) The minimal distance of the generating lines and the *I S A* is $\theta^*=$ constant,

b) Under a constant angle θ the generating lines of this hyperboloid intersects the generators of a cylinder of constant radius θ^* and its axis is the *I S A*.

Definition (2.4.1): (i) If the generating lines of a ruled surface have a constant angle with a definite line then the ruled surface is called an constant inclination ruled surface.

(ii) A ruled surface is called as a constant axis ruled surface if not only the inclination angle between the ruling and a definite

line is always constant, but also the minimal distance between these two lines is constant.

According to this definition a hyperboloid of one-sheet is a constant axis ruled surface. On the other hand the minimal distance of the axis $I\ S\ A$ of the cylinder and the generators of the hyperboloid is constant. Therefore, this cylinder is the envelope of the generators. Hence, by Study's map we may state that: The spatial equivalent of a circle on dual unit sphere is, in generally, a one-parameter family of hyperboloid of one-sheet. The intersection of each hyperboloid of one-sheet and the corresponding plane $l_1 = \varphi^*$ is the circle $l_2^2 + l_3^2 = c_2^2$. Therefore the envelope is a right circular cylinder.

Special Cases

1) If $\theta = \frac{\pi}{2}$ and $\theta^* \neq 0$, then from equations (2.4.5), (2.4.6) and (2.4.8) we obtain, respectively:

$$\delta = h, \quad \gamma = 0, \quad \theta^* + \Gamma = 0, \tag{2.4.10}$$

and

$$\left.\begin{aligned} l_1 &= \varphi^*, \\ l_2 \cos\varphi - l_3 \sin\varphi &= \theta^*. \end{aligned}\right\} \tag{2.4.11}$$

The equations (2.4.11) give ruled surfaces which are the intersections of the planes $l_1 = \varphi^*$ and the hyperbolic paraboloids:

$$(l_3 + \frac{\vartheta^*}{\sin\varphi})^2 = l_2^2 \cot^2\varphi. \qquad (2.4.12)$$

2) If $\theta = 0$ (or π) and $\theta^* \neq 0$, in this case δ and γ are undefined, and equations (2.4.8) reduces to

$$\left.\begin{aligned} l_1 &= \mu, \\ l_2 &= -\theta^* \sin\varphi, \\ l_3 &= \theta^* \cos\varphi. \end{aligned}\right\} \qquad (2.4.13)$$

or

$$l_2^2 + l_3^2 = {\theta^*}^2, \quad l_1 = \mu, \qquad (2.4.14)$$

which represents a right circular cylinder.

3) If $0 < \theta < \frac{\pi}{2}$ and $\theta^* = 0$, then from equations (2.4.5), (2.4.6) and (2.4.8) we have:

$$\gamma\delta - \Gamma = 0, \qquad (2.4.15)$$

and

$$\left.\begin{aligned} l_1 &= \varphi^* \sin^2\theta + \mu\cos\theta, \\ l_2 &= -(\varphi^* \cos\theta - \mu)\sin\theta\cos\varphi, \\ l_3 &= -(\varphi^* \cos\theta - \mu)\sin\theta\sin\varphi. \end{aligned}\right\} \qquad (2.4.16)$$

By eliminating μ, the ruled surface is:

$$l_2^2 + l_3^2 = (\frac{L_1}{\cot\,\varphi})^2. \qquad (2.4.17)$$

4) If $\theta = \frac{\pi}{2}$ and $\theta^* = 0$, then equations (2.4.5), (2.4.6) and (2.4.8) gives us:

$$\delta = h, \quad \gamma = \Gamma = 0, \qquad (2.4.18)$$

and

$$\left.\begin{aligned} l_1 &= \varphi^*, \\ l_2 &= \mu\cos\theta, \\ l_3 &= \mu\sin\varphi. \end{aligned}\right\} \qquad (2.4.19)$$

In this case the dual circle C is a great circle on K_f. Then from equation (2.4.19) the ruled surface is

$$l_1 = h\tan^{-1}(\frac{l_3}{l_2}), \qquad (2.4.20)$$

which represents a right helicoid..

Hence, by Study's map we may state that: The spatial equivalent of a great circle on dual unit sphere in the dual 3-space D^3 is, in generally, a right helicoid.

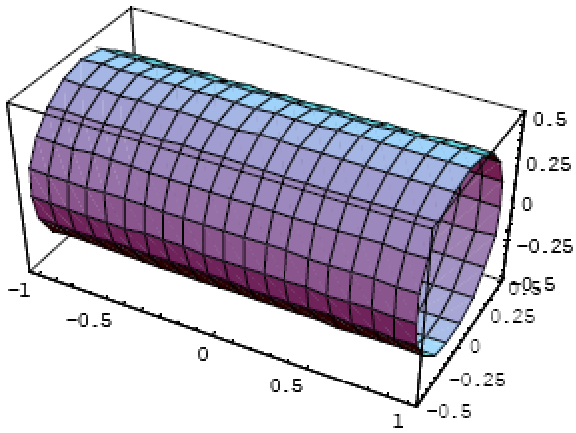

Figure (2.4.1): Ruled surface in the domain:

$\varphi \in [0,2\pi], \mu \in [-1,1], \theta = 0(or\ \pi), \theta^* = \frac{1}{2}, h = \frac{1}{2}.$

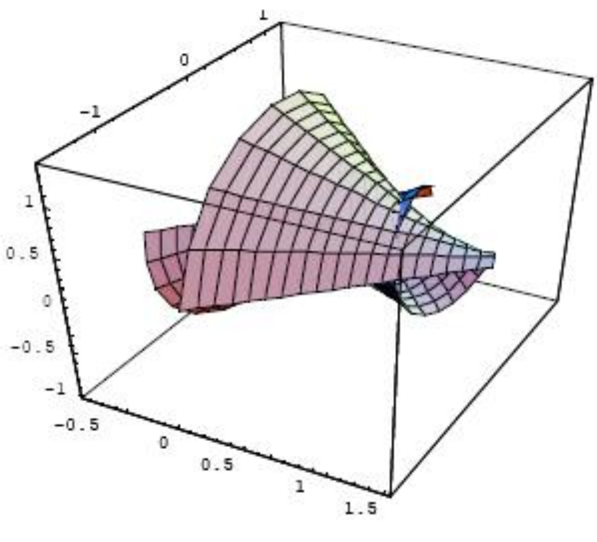

Figure (2.4.2): Ruled surface in the domain:

$\varphi \in [0,2\pi], \mu \in [-1,1], \theta = \frac{\pi}{4}, \theta^* = \frac{1}{4}, h = \frac{1}{4}.$

CHAPTER 3

Differential Geometric Conditions Between Curves and Ruled Surfaces

In this chapter , a system of differential equation determining ruled surface is introduced using the invariants of a given space curve in E^3. In the special cases the solutions of the this system of differential equations are obtained.

(3.1) Space Curves and Ruled Surfaces

Now, let the space curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be written as:

$$\boldsymbol{\alpha}(s)= l(s)\mathbf{t}(s)+m(s)\mathbf{n}(s)+h(s)\mathbf{b}(s). \qquad (3.1.1)$$

where $l(s)$, $m(s)$ and $h(s)$ are scalar differentiable functions. In such case the Blaschke frame is defined as:

$$\left.\begin{aligned} \mathbf{E}_1(s)&= \mathbf{t}(s)+\varepsilon(\boldsymbol{\alpha}(s)\times \mathbf{t}(s)),\\ \mathbf{E}_2(s)&= \mathbf{n}(s)+\varepsilon(\boldsymbol{\alpha}(s)\times \mathbf{n}(s)),\\ \mathbf{E}_3(s)&= \mathbf{b}(s)+\varepsilon(\boldsymbol{\alpha}(s)\times \mathbf{b}(s)). \end{aligned}\right\} \qquad (3.1.2)$$

By using the properties of vectorial products and from equations (3.1.1) and (3.1.2), we can express these vectors in the matrix form as follows:

$$\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix} = \begin{bmatrix}1 & \varepsilon h & -\varepsilon m\\ -\varepsilon h & 1 & \varepsilon l\\ \varepsilon m & -\varepsilon l & 1\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}. \tag{3.1.3}$$

Equation (3.1.3) says that E. Study mapping corresponds with a dual orthogonal matrix.

If we find $l = l(s)$, $m = m(s)$ and $h = h(s)$, we can determine the ruled surfaces $\mathbf{E}_i = \mathbf{E}_i(s)$, $(i\ =1,2,3)$ in terms of the invariants of the given curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$.

If the equation (3.1.1) is differentiated with respect to s and making use of the derivative formulas (1.1.4), we obtain:

$$\frac{d\boldsymbol{\alpha}}{ds} = \mathbf{t} = (l^{'} - m\kappa)\mathbf{t} + (m^{'} + l\kappa - h\tau)\mathbf{n} + (h^{'} + m\tau)\mathbf{b},$$

or the system of differential equation

$$\left.\begin{aligned} l^{'} &= m\kappa + 1,\\ m^{'} &= -l\kappa + h\tau,\\ h^{'} &= -m\tau. \end{aligned}\right\} \tag{3.1.4}$$

In the system (3.1.4), eliminating h, m and their derivatives we get the differential equation of the third order with variable coefficients

$$\left[\frac{1}{\tau}\left[\frac{1}{\kappa}\left[(l^{'} - 1)\right]^{'} + l\kappa\right]\right]^{'} + \frac{\tau}{\kappa}(l^{'} - 1) = 0. \tag{3.1.5}$$

Since $l = l(s)$ is the solution of the differential equation (3.1.5) and the functions $m = m(s)$ and $h = h(s)$ can be determined from (3.1.4), the ruled surfaces $\mathbf{E}_i = \mathbf{E}_i(s)$ are obtained in terms of the invariants of the curve.

The general solution of (3.1.5) can not be found but it can be solved in the special cases. Now let us solve it in the special cases.

1- If $l = 0$, from the system (3.1.4), we have:

$$\left.\begin{aligned} m\kappa + 1 &= 0, \\ m' &= h\tau, \\ h' &= -m\tau. \end{aligned}\right\} \tag{3.1.6}$$

Then, from the equations $(3.1.6)_1$ and $(3.1.6)_2$, the functions m and h should be in the form

$$\left.\begin{aligned} m &= -\frac{1}{\kappa}, \\ h &= -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'. \end{aligned}\right\} \tag{3.1.7}$$

And these functions must satisfy the equation $(3.1.6)_3$. Hence the relation

$$\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' + \frac{\tau}{\kappa} = 0, \tag{3.1.8}$$

exists between the invariants κ and τ of the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$.

The equation (3.1.8) shows that, the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ is a spherical curve and the condition $l = 0$ is equivalent to $l^2 + m^2 + h^2 = r^2$ where $r \in \mathbb{R}$. In this case, the vector $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ has the following form with the aid of the equation (3.1.1)

$$\boldsymbol{\alpha} = -\frac{1}{\kappa}\mathbf{n} - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{b}. \qquad (3.1.9)$$

Blaschke vectors of the ruled surface $\mathbf{E}_1 = \mathbf{E}_1(s)$ are determined by the dual-unit vectors from (3.1.2) and (3.1.9) as:

$$\left.\begin{aligned} \mathbf{E}_1 &= \mathbf{t} + \varepsilon\left[-\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{n} + \frac{1}{\kappa}\mathbf{b}\right], \\ \mathbf{E}_2 &= \mathbf{n} + \varepsilon\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{t}, \\ \mathbf{E}_3 &= \mathbf{b} - \varepsilon\frac{1}{\kappa}\mathbf{t}. \end{aligned}\right\} \qquad (3.1.10)$$

If the formulas $(3.1.10)_1$ and $(3.1.10)_3$ are differentiated with respect to s and using the equations (1.1.4), (2.3.5) and (2.3.6), then the real and the dual parts of obtained expressions found, respectively, are:

$$\left.\begin{aligned} p = \kappa, \quad p^* &= -\left[\frac{\tau}{\kappa} + \left[\frac{1}{\tau}\left[\frac{1}{\kappa}\right]'\right]'\right], \\ q = \tau, \quad q^* &= 1. \end{aligned}\right\} \qquad (3.1.11)$$

In accordance with (3.1.8) (or (2.3.6)), we have $p^* = 0$ which shows that the ruled surface $\mathbf{E}_1(s)$ is a tangential tourse.

2- If $m = 0$, the system

$$\left.\begin{aligned} l' &= 1, \\ -l\kappa + h\tau &= 0, \\ h' &= 0, \end{aligned}\right\} \qquad (3.1.12)$$

is found from the equations (3.1.4). Solving the first and third equations of (3.1.12) we get:

$$l = s + c_1, \quad h = c_2. \qquad (3.1.13)$$

where c_1 and c_2 are real constants. Since the second equation of (3.1.12) must be satisfied by the equations (3.1.13), the relation

$$\frac{\kappa}{\tau} = \frac{c_2}{s + c_1} \qquad (3.1.14)$$

exists between the invariants and of the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. In that case, the vector $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ can be written with the aid of equation (3.1.1) as:

$$\boldsymbol{\alpha} = (s + c_1)\mathbf{t} + c_2\mathbf{b} \qquad (3.1.15)$$

Blaschke vectors of the ruled surface are also determined by the dual unit vectors from (3.1.2) and (3.1.15) as the following:

$$\left.\begin{aligned} \mathbf{E}_1 &= \mathbf{t} + \varepsilon c_2 \mathbf{n}, \\ \mathbf{E}_2 &= \mathbf{n} + \varepsilon[-c_2\mathbf{t} + (s + c_1)]\mathbf{b}, \\ \mathbf{E}_3 &= \mathbf{b} - \varepsilon(s + c_1)\mathbf{n}. \end{aligned}\right\} \tag{3.1.16}$$

If the equations $(3.1.16)_1$ and $(3.1.16)_3$ are differentiated with respect to s and using (1.1.4), (2.3.5) and (2.3.6), then the real and dual parts of obtained expressions are found, respectively,

$$\left.\begin{aligned} p &= \kappa, \quad p^* = 0, \\ q &= \tau, \quad q^* = 0. \end{aligned}\right\} \tag{3.1.17}$$

In this case, the ruled surface $\mathbf{E}_1(s)$ becomes a cone and $q^* = 0$ shows that the ruled surface $\mathbf{E}_3(s)$ generated by the dual unit vector $\mathbf{E}_3 = \mathbf{E}_3(s)$ is also a cone.

3- If $h = 0$, the curve becomes a plane curve. Because the vector $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ lies in the osculating plane of the curve. Thus, from the system of the system of the equation (3.1.4), we have:

$$\left.\begin{aligned} l' &= m\kappa + 1, \\ m' &= -l\kappa, \\ \tau &= 0. \end{aligned}\right\} \tag{3.1.18}$$

When m and its derivative are eliminated from (3.1.18), the following differential equation is found:

$$l'' - \frac{\kappa'}{\kappa} l' + \kappa^2 l = \frac{\kappa'}{\kappa}, \tag{3.1.19}$$

If we make the parameter change like this $t = \int_0^s \kappa ds$ in (3.1.19), the differential equation with constant coefficients at

$$\frac{d^2 l}{dt^2} + l = \frac{\dot{\kappa}}{\kappa^2}, \quad \dot{\kappa} = \frac{d\kappa}{dt}. \tag{3.1.20}$$

The solution of this equation is

$$l = \mathrm{c}_1 \cos \int_0^s \kappa ds + \mathrm{c}_2 \sin \int_0^s \kappa ds + \left(\frac{1}{\mathrm{D}^2+1}\right)\left(\frac{\dot{\kappa}}{\kappa^2}\right). \tag{3.1.21}$$

where c_1 and c_2 are constants and D denotes the derivative operator as $\mathrm{D} = \frac{d}{dt}$. From $m = \frac{1}{\kappa}(l' - 1)$, we have:

$$m = -\mathrm{c}_1 \sin \int_0^s \kappa ds + \mathrm{c}_2 \cos \int_0^s \kappa ds + \frac{1}{\kappa}\left[\left(\frac{\mathrm{D}}{\mathrm{D}^2+1}\right)\left(\frac{\dot{\kappa}}{\kappa^2}\right) - 1\right]. \tag{3.1.22}$$

When the plane curve is a circle ($\kappa = const.$), the equality $\int_0^s \kappa ds = 2\pi$ exists. Thus, from the equations (3.1.21) and (3.1.22) the functions $l = l(s)$ and $m = m(s)$ are obtained as:

$$\left.\begin{aligned} l &= \mathrm{c}_1, \\ m &= \mathrm{c}_2 - \frac{1}{\kappa}. \end{aligned}\right\} \tag{3.1.23}$$

Since the equations (3.1.23) must satisfy (3.1.18), there is the condition $c_1 = c_2 = 0$ so that

$$\left.\begin{aligned} l &= 0, \\ m &= -\frac{1}{\kappa}. \end{aligned}\right\} \tag{3.1.24}$$

By substituting (3.1.24) into (3.1.2), then Blaschke vectors of the ruled surface $\mathbf{E}_1(s)$ are determined by the dual unit vectors as:

$$\left.\begin{aligned} \mathbf{E}_1 &= \mathbf{t} + \varepsilon \frac{1}{\kappa} \mathbf{b}, \\ \mathbf{E}_2 &= \mathbf{n}, \\ \mathbf{E}_3 &= \mathbf{b} - \varepsilon \frac{1}{\kappa} \mathbf{t}. \end{aligned}\right\} \tag{3.1.25}$$

If the equations $(3.1.25)_1$ and $(3.1.25)_3$ are differentiated with respect to s and making use of the equations (1.1.4), (2.3.5) and (2.3.6) , then the real and dual parts of obtained expressions are found respectively.

$$\left.\begin{aligned} p = \kappa = const., \qquad p^* &= \pm 1, \\ q = 0, \qquad q^* &= 1. \end{aligned}\right\} \tag{3.1.26}$$

Moreover, from (2.1.18) distribution parameter of the ruled surface $\mathbf{E}_1(s)$ becomes

$$\delta = \pm \frac{1}{\kappa} = const. \tag{3.1.27}$$

4- Finally, let the curve $\boldsymbol{\alpha}(s)$ be a circular helix. In this case, τ and κ are constants. When we eliminate l and h from $(3.1.4)_2$, the differential equation is obtained:

$$m'' + (\kappa^2 + \tau^2)m = -\kappa. \qquad (3.1.28)$$

If we make the parameter change like

$$t = \int_0^s \sqrt{\kappa^2 + \tau^2}\, ds,$$

in (3.1.28), the differential equation is obtained:

$$\frac{d^2 m}{dt^2} + m = -\left(\frac{\kappa}{\kappa^2+\tau^2}\right).$$

Then, its solution is

$$m = \mathrm{c}_1 \cos\sqrt{\kappa^2 + \tau^2}\, s + \mathrm{c}_2 \sin\sqrt{\kappa^2 + \tau^2}\, s - \left(\frac{1}{D^2+1}\right)\left(\frac{\kappa}{\kappa^2+\tau^2}\right), \qquad (3.1.29)$$

and from the equations $(3.1.4)_1$ and $(3.1.4)_3$, we have l and h as:

$$l = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\left[\mathrm{c}_1 \sin\sqrt{\kappa^2 + \tau^2}\, s - \mathrm{c}_2 \cos\sqrt{\kappa^2 + \tau^2}\, s + \frac{\tau^2 s}{\kappa\sqrt{\kappa^2+\tau^2}}\right], \qquad (3.1.30)$$

$$h = \frac{-\tau}{\sqrt{\kappa^2+\tau^2}}\left[\mathrm{c}_1 \sin\sqrt{\kappa^2 + \tau^2}\, s - \mathrm{c}_2 \cos\sqrt{\kappa^2 + \tau^2}\, s - \frac{\kappa s}{\sqrt{\kappa^2+\tau^2}}\right]. \qquad (3.1.31)$$

If the last three equations are substituted into the equation (3.1.3), then the ruled surface $\mathbf{E}_1(s)$ is determined. Making others calculations, the invariants of the ruled surface $\mathbf{E}_1(s)$ are found as the following:

$$\left.\begin{aligned} p = \kappa = const., \qquad & p^* = 0, \\ q = \tau = const., \qquad & q^* = 1. \end{aligned}\right\} \tag{3.1.32}$$

Thus, the ruled surface $\mathbf{E}_1(s)$ becomes a developable surface with constant inclination because of $\frac{q}{p} = \frac{\tau}{\kappa} = constant$ and $p^* = 0$.

(3.2) Strip Curves and Ruled Surfaces

Now, Let $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \longrightarrow M \subseteq E^3$ be an arc-length parameterized curve on a surface *M*. In this case $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ is called strip curve on *M* and we have the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and Darboux frame $\{\mathbf{t}, \mathbf{u}, \mathbf{v}\}$ along the curve, where $\mathbf{t}$ is the unit tangent of the curve and $\mathbf{u}$ is the unit normal of the surface *M*, and $\mathbf{v} = \mathbf{t} \times \mathbf{u}$. Moreover, the relationships between these frames are as follows [10]:

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{3.2.1}$$

By differentiating (3.2.1) and using Frenet formula, we obtain

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{u}\\ \mathbf{v}\end{bmatrix}=\begin{bmatrix}0 & \kappa_n & \kappa_g\\ -\kappa_n & 0 & \tau_g\\ -\kappa_g & -\tau_g & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{u}\\ \mathbf{v}\end{bmatrix}, \qquad (3.2.2)$$

where

$$\kappa_n = \kappa\cos\theta,\ \kappa_g = \kappa\sin\theta \ \text{ and } \ \tau_g = \tau\sin\theta. \qquad (3.2.3)$$

The symbols κ_n is the normal curvature, κ_g is the geodesic curvature and τ_g the geodesic torsion of the curve [10].

As a similar way to section (3.1), we may write:

$$\boldsymbol{\alpha}(s)= l(s)\mathbf{t} + m(s)\mathbf{u} + h(s)\mathbf{v}. \qquad (3.2.4)$$

Hence, the Blaschke frame in the matrix form as:

$$\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}=\begin{bmatrix}1 & \varepsilon h & -\varepsilon m\\ -\varepsilon h & 1 & \varepsilon l\\ \varepsilon m & -\varepsilon l & 1\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{u}\\ \mathbf{v}\end{bmatrix}. \qquad (3.2.5)$$

If the equation (3.2.4) is differentiated with respect to s, and making use of the derivative formulas (3.2.2), are obtain:

$$\left.\begin{aligned} l' - m\kappa_n - h\kappa_g &= 1,\\ l\kappa_n + m' - h\tau_g &= 0,\\ l\kappa_g + m\tau_g + h' &= 0.\end{aligned}\right\} \qquad (3.2.6)$$

From (3.2.6) we discuss many different cases:

A- The strip curve be asymptotic strip $(\kappa_n = 0)$. In this case, from equations (3.2.6) we get:

$$\left.\begin{aligned} l' - h\kappa_g &= 1, \\ m' - h\tau_g &= 0, \\ l\kappa_g + m\tau_g + h' &= 0. \end{aligned}\right\} \tag{3.2.7}$$

(a)- If $l(s)= 0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s), \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.7)

$$\left.\begin{aligned} h + \frac{1}{\kappa_g} &= 0, \\ m' - h\tau_g &= 0, \\ m\tau_g + h' &= 0. \end{aligned}\right\} \tag{3.2.8}$$

are found. If we make change the parameter as $d\beta = \tau_g ds$ into the last two equations of (3.2.8), then we obtain:

$$\frac{d^2m}{d\beta^2} + m = 0. \tag{3.2.9}$$

Equation (3.2.9) has the solution

$$m(s) = c_1 \cos \int_0^s \tau_g \, ds + c_2 \sin \int_0^s \tau_g \, ds, \tag{3.2.10}$$

where c_1 and c_2 are real constants. From (3.2.8), it is clear that

$$h(s) = -c_1 \sin \int_0^s \tau_g ds + c_2 \cos \int_0^s \tau_g \, ds. \qquad (3.2.11)$$

Therefore, from (3.2.5) the Blaschke frame:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s) + \varepsilon\Big\{\Big[-c_1 \sin \int_0^s \tau_g ds + c_2 \cos \int_0^s \tau_g \, ds\Big]\mathbf{u} \\ &\qquad -\Big[c_1 \cos \int_0^s \tau_g \, ds + c_2 \sin \int_0^s \tau_g ds\Big]\mathbf{v}\Big\}, \\ \mathbf{E}_2(s) &= \mathbf{u}(s) - \varepsilon\Big[-c_1 \sin \int_0^s \tau_g ds + c_2 \cos \int_0^s \tau_g \, ds\Big]\mathbf{t}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s) + \varepsilon\Big[c_1 \cos \int_0^s \tau_g \, ds + c_2 \sin \int_0^s \tau_g ds\Big]\mathbf{t}. \end{aligned}\right\} \qquad (3.2.12)$$

(b)- If $m(s) = 0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s),\ \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.7)

$$\left.\begin{aligned} l' - h\kappa_g &= 1, \\ -h\tau_g &= 0, \\ l\kappa_g + h' &= 0. \end{aligned}\right\} \qquad (3.2.13)$$

are found from the system (3.2.7). If $h \neq 0$ ($\tau_g = 0$) and by making use of change the parameter as $\beta = \int_0^s \kappa_g ds$ into (3.2.13), then we have

$$\left.\begin{aligned} \frac{dl}{d\beta} - h &= \frac{1}{\kappa_g}, \\ l + \frac{dh}{d\beta} &= 0. \end{aligned}\right\} \qquad (3.2.14)$$

From the last two equations, we get:

$$\frac{d^2h}{d\beta^2} + h = -\rho_g; \qquad \left(\rho_g = \frac{1}{\kappa_g}\right). \tag{3.2.15}$$

of course, we have

$$h_c = \mathrm{A}\cos\beta + \mathrm{B}\sin\beta, \tag{3.2.16}$$

Let us seek a particular solution by variation of parameters. Put

$$h_p = \mathrm{c}_1\cos\beta + \mathrm{c}_2\sin\beta, \tag{3.2.17}$$

from which

$$h_p' = -\mathrm{c}_1\sin\beta + \mathrm{c}_2\cos\beta + \mathrm{c}_1'\cos\beta + \mathrm{c}_2'\sin\beta,$$

where $\frac{dh_p}{d\beta} = h_p'$. Next set

$$\mathrm{c}_1'\cos\beta + \mathrm{c}_2'\sin\beta = 0. \tag{3.2.18}$$

so that

$$h_p' = -\mathrm{c}_1\sin\beta + \mathrm{c}_2\cos\beta.$$

Then

$$h_p'' = -\mathrm{c}_1\cos\beta - \mathrm{c}_2\sin\beta - \mathrm{c}_1'\sin\beta + \mathrm{c}_2'\cos\beta. \tag{3.2.19}$$

Next we eliminate h_p by combining equations (3.2.17), (3.2.19) with the original equation (3.2.15). Thus, we get the relation

$$-c_1' \sin\beta + c_2' \cos\beta = -\rho_g. \qquad (3.2.20)$$

From (3.2.20) and (3.2.18), c_1' is easily eliminated. The result is

$$c_2' = -\rho_g \cos\beta,$$

so that

$$c_2 = -\int \rho_g \cos\beta \, d\beta. \qquad (3.2.21)$$

From (3.2.20) and (3.2.18) it also follows easily that

$$c_1' = \rho_g \sin\beta,$$

or

$$c_1 = \int \rho_g \sin\beta \, d\beta. \qquad (3.2.22)$$

Returning to equation (3.2.17) with known c_1 from (3.2.22) and c_2 from (3.2.21), the particular solution is

$$h_p = \left(\int \rho_g \sin\beta \, d\beta\right) \cos\beta - \left(\int \rho_g \cos\beta \, d\beta\right) \sin\beta. \qquad (3.2.23)$$

Then, from equations (3.2.16) and (3.2.23), the general solution of (3.2.15) is

$$h = \mathrm{A} \cos\beta + \mathrm{B} \sin\beta + \left(\int \rho_g \sin\beta \, d\beta\right) \cos\beta$$
$$-\left(\int \rho_g \cos\beta \, d\beta\right) \sin\beta. \qquad (3.2.24)$$

In particular if $\kappa_g = const.$, then $\beta = \kappa_g s$ and from equations (3.2.21) and (3.2.22), we obtain:

$$\left.\begin{aligned} c_2 &= -\frac{1}{\kappa_g}\sin(\kappa_g s), \\ c_1 &= -\frac{1}{\kappa_g}\cos(\kappa_g s). \end{aligned}\right\} \tag{3.2.25}$$

Therefore, the general solution of (3.2.24) is

$$h(s) = \mathrm{A}\cos(\kappa_g s) + \mathrm{B}\sin(\kappa_g s) - \frac{1}{\kappa_g}. \tag{3.2.26}$$

And from equations (3.2.14), we find

$$l(s) = \mathrm{A}\sin(\kappa_g s) - \mathrm{B}\cos(\kappa_g s). \tag{3.2.27}$$

Therefore, from (3.2.5) the Blaschke frame:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s) + \varepsilon\left\{\left[\mathrm{A}\cos(\kappa_g s) + \mathrm{B}\sin(\kappa_g s) - \frac{1}{\kappa_g}\right]\mathbf{u}\right\}, \\ \mathbf{E}_2(s) &= \mathbf{u}(s) + \varepsilon\left\{\left[-\left(\mathrm{A}\cos(\kappa_g s) + \mathrm{B}\sin(\kappa_g s) - \frac{1}{\kappa_g}\right)\right]\mathbf{t}\right. \\ &\qquad \left. + \left[\mathrm{A}\sin(\kappa_g s) - \mathrm{B}\cos(\kappa_g s)\right]\mathbf{v}\right\}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s) - \varepsilon\left\{\left[\mathrm{A}\sin(\kappa_g s) - \mathrm{B}\cos(\kappa_g s)\right]\mathbf{u}\right\}. \end{aligned}\right\} \tag{3.2.28}$$

(c)- If $h(s) = 0$, the strip curve is located in the vector space $\mathrm{S}_p\{\mathbf{u}(s), \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.7)

$$\left.\begin{aligned} l' &= 1, \\ m' &= 0, \\ l\kappa_g + m\tau_g &= 0. \end{aligned}\right\} \tag{3.2.29}$$

are found. Then the solutions, we obtain

$$l(s) = s + c_1, \tag{3.2.30}$$

$$m(s) = c_2. \tag{3.2.31}$$

where c_1 and c_2 are real constants. Therefore, from equations (3.2.30), (3.2.31) and (3.2.5) the Blaschke frame:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s) - \varepsilon c_2 \mathbf{v}, \\ \mathbf{E}_2(s) &= \mathbf{u}(s) + \varepsilon[s + c_1]\mathbf{v}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s) + \varepsilon\{c_2\mathbf{t} - [s + c_1]\mathbf{u}\}. \end{aligned}\right\} \tag{3.2.32}$$

B- The strip curve be curvature strip ($\tau_g = 0$). In this case, from equations (3.2.6) we get:

$$\left.\begin{aligned} l' - m\kappa_n - h\kappa_g &= 1, \\ l\kappa_n + m' &= 0, \\ l\kappa_g + h' &= 0. \end{aligned}\right\} \tag{3.2.33}$$

(a)- If $l(s)= 0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s), \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.33)

$$\left.\begin{aligned} -m\kappa_n - h\kappa_g &= 1, \\ m' &= 0, \\ h' &= 0. \end{aligned}\right\} \qquad (3.2.34)$$

are found. Then the solutions, we obtain

$$m(s)= c_1, \qquad (3.2.35)$$

$$h(s)= c_2. \qquad (3.2.36)$$

where c_1 and c_2 are real constants. Therefore, from equations (3.2.35), (3.2.36) and (3.2.5) the Blaschke frame:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s)+\varepsilon[c_2\mathbf{u} - c_1\mathbf{v}], \\ \mathbf{E}_2(s) &= \mathbf{u}(s)-\varepsilon c_2\mathbf{t}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s)+\varepsilon c_1\mathbf{t}. \end{aligned}\right\} \qquad (3.2.37)$$

(b)- If $m(s)= 0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s), \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.33)

$$\left.\begin{aligned} l' - h\kappa_g &= 1, \\ l\kappa_n &= 0, \\ l\kappa_g + h' &= 0. \end{aligned}\right\} \qquad (3.2.38)$$

are found from the system (3.2.33). If $l \neq 0$ ($\kappa_n = 0$) and by making use of change the parameter as $\beta = \int_0^s \kappa_g ds$ into (3.2.38), then we have

$$\left.\begin{aligned} \frac{dl}{d\beta} - h &= \frac{1}{\kappa_g}, \\ l + \frac{dh}{d\beta} &= 0. \end{aligned}\right\} \qquad (3.2.39)$$

From the last two equations, we get:

$$\frac{d^2h}{d\beta^2} + h = -\rho_g; \qquad (\rho_g = \frac{1}{\kappa_g}). \qquad (3.2.40)$$

Likewise, we have

$$h_c = \mathrm{A} \cos\beta + \mathrm{B} \sin\beta, \qquad (3.2.41)$$

Moreover, the particular solution is

$$h_p = \mathrm{c}_1 \cos\beta + \mathrm{c}_2 \sin\beta, \qquad (3.2.42)$$

where

$$\mathrm{c}_1 = \int \rho_g \sin\beta \, d\beta,$$

$$\mathrm{c}_2 = -\rho_g \int \cos\beta \, d\,\beta.$$

In particular if $\kappa_g = const.$, then we have:

$$\mathrm{c}_1 = -\frac{1}{\kappa_g}\cos(\kappa_g s), \tag{3.2.43}$$

and

$$\mathrm{c}_2 = -\frac{1}{\kappa_g}\sin(\kappa_g s). \tag{3.2.44}$$

Returning to equation (3.2.42) with known c_1 from (3.2.43) and c_2 from (3.2.44), are write the particular solution

$$h_p = \left(-\frac{1}{\kappa_g}\cos(\kappa_g s)\right)\cos\beta + \left(-\frac{1}{\kappa_g}\sin(\kappa_g s)\right)\sin\beta.$$

Hence, the general solution of (3.2.40) is

$$h = h_c + h_p,$$

or

$$h(s) = \mathrm{A}\cos(\kappa_g s) + \mathrm{B}\sin(\kappa_g s) - \frac{1}{\kappa_g}. \tag{3.2.45}$$

From equation (3.2.39), we have

$$l(s) = \mathrm{A}\sin(\kappa_g s) - \mathrm{B}\cos(\kappa_g s). \tag{3.2.46}$$

Similarly, from (3.2.5) the Blaschke frame:

$$\left.\begin{aligned}
\mathbf{E}_1(s) &= \mathbf{t}(s)+\varepsilon\left\{\left[\mathrm{A}\cos(\kappa_g s)+\mathrm{B}\sin(\kappa_g s)-\frac{1}{\kappa_g}\right]\mathbf{u}\right\},\\
\mathbf{E}_2(s) &= \mathbf{u}(s)+\varepsilon\left\{\left[-\left(\mathrm{A}\cos(\kappa_g s)+\mathrm{B}\sin(\kappa_g s)-\frac{1}{\kappa_g}\right)\right]\mathbf{t}\right.\\
&\qquad \left.+\left[\mathrm{A}\sin(\kappa_g s)-\mathrm{B}\cos(\kappa_g s)\right]\mathbf{v}\right\},\\
\mathbf{E}_3(s) &= \mathbf{v}(s)-\varepsilon\left\{\left[\mathrm{A}\sin(\kappa_g s)-\mathrm{B}\cos(\kappa_g s)\right]\mathbf{u}\right\}.
\end{aligned}\right\} \quad (3.2.47)$$

(c)- If $h(s)=0$, the strip curve is located in the vector space $\mathrm{S}_p\{\mathbf{u}(s),\ \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.33)

$$\left.\begin{aligned}
l' - m\kappa_n &= 1,\\
l\kappa_n + m' &= 0,\\
l\kappa_g &= 0.
\end{aligned}\right\} \quad (3.2.48)$$

are found from the system (3.2.33). If $l \neq 0$ ($\kappa_g = 0$) and by making use of change the parameter as $\beta = \int_0^s \kappa_n ds$ into (3.2.48), then we have

$$\left.\begin{aligned}
\frac{dl}{d\beta} - m &= \frac{1}{\kappa_n},\\
l + \frac{dm}{d\beta} &= 0.
\end{aligned}\right\} \quad (3.2.49)$$

From the last two equations, we get:

$$\frac{d^2 m}{d\beta^2} + m = -\rho_n; \qquad \left(\rho_n = \frac{1}{\kappa_n}\right). \tag{3.2.50}$$

Similarly, we have

$$m_c = \mathrm{A}\cos\beta + \mathrm{B}\sin\beta, \tag{3.2.51}$$

Also, the particular solution is

$$m_p = \mathrm{c}_1 \cos\beta + \mathrm{c}_2 \sin\beta, \tag{3.2.52}$$

where

$$\mathrm{c}_1 = \int \rho_n \sin\beta \, d\beta,$$

$$\mathrm{c}_2 = -\rho_n \int \cos\beta \, d\,\beta.$$

In particular if $\kappa_n = const.$, then we have:

$$\mathrm{c}_1 = -\frac{1}{\kappa_n}\cos(\kappa_n s), \tag{3.2.53}$$

and

$$\mathrm{c}_2 = -\frac{1}{\kappa_n}\sin(\kappa_n s). \tag{3.2.54}$$

Returning to equation (3.2.52) with known c_1 from (3.2.53) and c_2 from (3.2.54), are write the particular solution

$$m_p = \left(-\frac{1}{\kappa_n}\cos(\kappa_n s)\right)\cos\beta + \left(-\frac{1}{\kappa_n}\sin(\kappa_n s)\right)\sin\beta.$$

Hence, the general solution of (3.2.50) is

$$m = m_c + m_p,$$

or

$$m(s) = \mathrm{A}\cos(\kappa_n s) + \mathrm{B}\sin(\kappa_n s) - \frac{1}{\kappa_n}. \qquad (3.2.55)$$

From equation (3.2.49), we have

$$l(s) = \mathrm{A}\sin(\kappa_n s) - \mathrm{B}\cos(\kappa_n s). \qquad (3.2.56)$$

As we did, the Blaschke frame in the form as:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s) - \varepsilon\left\{\left[\mathrm{A}\cos(\kappa_n s) + \mathrm{B}\sin(\kappa_n s) - \frac{1}{\kappa_n}\right]\mathbf{v}\right\}, \\ \mathbf{E}_2(s) &= \mathbf{u}(s) + \varepsilon\{[\mathrm{A}\sin(\kappa_n s) - \mathrm{B}\cos(\kappa_n s)]\mathbf{v}\}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s) + \varepsilon\left\{\left[\mathrm{A}\cos(\kappa_n s) + \mathrm{B}\sin(\kappa_n s) - \frac{1}{\kappa_n}\right]\mathbf{t}\right. \\ &\qquad \left. - [\mathrm{A}\sin(\kappa_n s) - \mathrm{B}\cos(\kappa_n s)]\mathbf{u}\right\}. \end{aligned}\right\} \qquad (3.2.57)$$

C- The strip curve be geodesic strip ($\kappa_g = 0$). In this case, from equations (3.2.6) we get:

$$\left.\begin{aligned} l' - m\kappa_n &= 1, \\ l\kappa_n + m' - h\tau_g &= 0, \\ m\tau_g + h' &= 0. \end{aligned}\right\} \qquad (3.2.58)$$

(a)- If $l(s)=0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s), \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.58)

$$\left.\begin{aligned} -m\kappa_n &= 1, \\ m' - h\tau_g &= 0, \\ m\tau_g + h' &= 0. \end{aligned}\right\} \tag{3.2.59}$$

are found. If we make change the parameter as $d\beta = \tau_g ds$ into the last two equations of (3.2.59), then we obtain:

$$\frac{d^2m}{d\beta^2} + m = 0. \tag{3.2.60}$$

Equation (3.2.60) has the solution

$$m(s) = c_1 \cos \int_0^s \tau_g \, ds + c_2 \sin \int_0^s \tau_g \, ds, \tag{3.2.61}$$

where c_1 and c_2 are real constants. From (3.2.59), it is clear that

$$h(s) = -c_1 \sin \int_0^s \tau_g \, ds + c_2 \cos \int_0^s \tau_g \, ds. \tag{3.2.62}$$

Therefore, the Blaschke frame in the form as:

$$\left.\begin{aligned}\mathbf{E}_1(s) &= \mathbf{t}(s)+\varepsilon\Big\{\Big[-c_1\sin\int_0^s \tau_g ds + c_2\cos\int_0^s \tau_g\, ds\Big]\mathbf{u}\\ &\qquad -\Big[c_1\cos\int_0^s \tau_g\, ds + c_2\sin\int_0^s \tau_g ds\Big]\mathbf{v}\Big\},\\ \mathbf{E}_2(s) &= \mathbf{u}(s)-\varepsilon\Big[-c_1\sin\int_0^s \tau_g ds + c_2\cos\int_0^s \tau_g\, ds\Big]\mathbf{t},\\ \mathbf{E}_3(s) &= \mathbf{v}(s)+\varepsilon\Big[c_1\cos\int_0^s \tau_g\, ds + c_2\sin\int_0^s \tau_g ds\Big]\mathbf{t}.\end{aligned}\right\} \quad (3.2.63)$$

(b)- If $m(s)=0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s),\ \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.58)

$$\left.\begin{aligned} l' &= 1,\\ l\kappa_n - h\tau_g &= 0,\\ h' &= 0.\end{aligned}\right\} \quad (3.2.64)$$

are found. Then the solutions, we obtain

$$l(s)= s + c_1, \quad (3.2.65)$$

$$h(s)= c_2. \quad (3.2.66)$$

where c_1 and c_2 are real constants. Likewise, the Blaschke frame in the form as:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s) + \varepsilon c_2 \mathbf{u}, \\ \mathbf{E}_2(s) &= \mathbf{u}(s) + \varepsilon\{-c_2\mathbf{t} + [s + c_1]\mathbf{v}\}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s) - \varepsilon[s + c_1]\mathbf{u}. \end{aligned}\right\} \qquad (3.2.67)$$

(c)- If $h(s) = 0$, the strip curve is located in the vector space $S_p\{\mathbf{u}(s), \mathbf{v}(s)\}$ at the point $\boldsymbol{\alpha}(s)$. In this situation, from the system (3.2.58)

$$\left.\begin{aligned} l' - m\kappa_n &= 1, \\ l\kappa_n + m' &= 0, \\ m\tau_g &= 0. \end{aligned}\right\} \qquad (3.2.68)$$

are found from the system (3.2.58). If $m \neq 0$ ($\tau_g = 0$) and by making use of change the parameter as $\beta = \int_0^s \kappa_n ds$ into (3.2.68), then we have

$$\left.\begin{aligned} \frac{dl}{d\beta} - m &= \frac{1}{\kappa_n}, \\ l + \frac{dm}{d\beta} &= 0. \end{aligned}\right\} \qquad (3.2.69)$$

From the last two equations, we get:

$$\frac{d^2m}{d\beta^2} + m = -\rho_n; \qquad (\rho_n = \frac{1}{\kappa_n}). \qquad (3.2.70)$$

Similarly, we have

$$m_c = \mathrm{A}\cos\beta + \mathrm{B}\sin\beta, \qquad (3.2.71)$$

Also, the particular solution is

$$m_p = \mathrm{c}_1 \cos\beta + \mathrm{c}_2 \sin\beta, \qquad (3.2.72)$$

where

$$\mathrm{c}_1 = \int \rho_n \sin\beta \, d\beta,$$

$$\mathrm{c}_2 = -\rho_n \int \cos\beta \, d\,\beta.$$

In particular if $\kappa_n = const.$, then we have:

$$\mathrm{c}_1 = -\frac{1}{\kappa_n}\cos(\kappa_n s), \qquad (3.2.73)$$

and

$$\mathrm{c}_2 = -\frac{1}{\kappa_n}\sin(\kappa_n s). \qquad (3.2.74)$$

Returning to equation (3.2.72) with known c_1 from (3.2.73) and c_2 from (3.2.74), are write the particular solution

$$m_p = \left(-\frac{1}{\kappa_n}\cos(\kappa_n s)\right)\cos\beta + \left(-\frac{1}{\kappa_n}\sin(\kappa_n s)\right)\sin\beta.$$

Hence, the general solution of (3.2.70) is

$$m = m_c + m_p,$$

or

$$m(s) = \mathrm{A}\cos(\kappa_n s) + \mathrm{B}\sin(\kappa_n s) - \frac{1}{\kappa_n}. \tag{3.2.75}$$

From equation (3.2.69), we have

$$l(s) = \mathrm{A}\sin(\kappa_n s) - \mathrm{B}\cos(\kappa_n s). \tag{3.2.76}$$

Therefore, from equation (3.2.5) the Blaschke frame:

$$\left.\begin{aligned} \mathbf{E}_1(s) &= \mathbf{t}(s) - \varepsilon\left\{\left[\mathrm{A}\cos(\kappa_n s) + \mathrm{B}\sin(\kappa_n s) - \frac{1}{\kappa_n}\right]\mathbf{v}\right\}, \\ \mathbf{E}_2(s) &= \mathbf{u}(s) + \varepsilon\{[\mathrm{A}\sin(\kappa_n s) - \mathrm{B}\cos(\kappa_n s)]\mathbf{v}\}, \\ \mathbf{E}_3(s) &= \mathbf{v}(s) + \varepsilon\left\{\left[\mathrm{A}\cos(\kappa_n s) + \mathrm{B}\sin(\kappa_n s) - \frac{1}{\kappa_n}\right]\mathbf{t}\right. \\ &\qquad \left. - [\mathrm{A}\sin(\kappa_n s) - \mathrm{B}\cos(\kappa_n s)]\mathbf{u}\right\}. \end{aligned}\right\} \tag{3.2.77}$$

CHAPTER 4

Bertrand offsets of ruled surface

In this chapter the notions of Bertrand offsets for ruled and developable surfaces are developed. Considering a ruled surface, like a general surface, as a two-parameter family of points, then its normal offset is a surface having common normals at all its points with the original base surface. Such surfaces will be referred to as ’parallel offsets’. Methods for the generation of both exact and approximate parallel offsets for certain class of surfaces have been developed by [18]. The same techniques can easily be specialized to the cases of ruled and developable surfaces. The results obtained, however, are not analogous to the theory of Bertrand curves and therefore do not carry the same intrinsic elegance and advantages.

(4.1) The scalar formulation

Here it is shown that if ruled surfaces are considered in the context of line geometry, as a one-parameter family of lines, then a theory analogous to the theory of Bertrand curves can be developed for such surfaces. In line geometry, the offset distance between two lines is defined in terms of a linear and angular offset. The linear offset ψ^* is the length of the common

perpendicular between the two lines. The angular offset ψ is the angle between the two lines measured in a plane orthogonal to the common perpendicular between the two lines. In the case of two curves which are Bertrand offsets of one another, the offset distance is taken along the normal to the base curve at each position of the generating point of the curve. For a ruled surface, however, at each position of the generator (or ruling), there are many normals to the surface along the generator. For instance, if one is not confined to ruled patches, then the normal to the surface at one infinite end of a ruling is parallel to the central tangent $\mathbf{e}_3$ of the surface. At the central or striction point on that same ruling, the normal to the surface rotates 90^{o} from the central tangent direction and becomes parallel to the central normal $\mathbf{e}_2$ of the surface. Tracing the same ruling, the normal to the surface rotates another 90^{o} and becomes parallel to $-\mathbf{e}_3$ at the other end of the ruling. With this
in mind, the following definition for Bertrand offset of a ruled surface is given.

Definition (4.1.1): Two ruled surfaces are said to be Bertrand offsets of one another if there exists a one-to-one correspondence between their rulings such that both surfaces have a common central normal at the striction points of their corresponding rulings.

If $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$ and $\bar{\mathbf{e}}_3$ are the Blaschke frame of the Bertrand offset $\bar{M}$, of M and $\bar{u}$ is the arc-length of the striction curve of $\bar{M}$, by interchanging the roles of M, we may write [31]:

$$\begin{bmatrix} \bar{\mathbf{e}}_1' \\ \bar{\mathbf{e}}_2' \\ \bar{\mathbf{e}}_3' \end{bmatrix} = \begin{bmatrix} 0 & \bar{p} & 0 \\ -\bar{p} & 0 & \bar{q} \\ 0 & -\bar{q} & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \\ \bar{\mathbf{e}}_3 \end{bmatrix}, \tag{4.1.1}$$

Call $\psi = \psi(u)$ the offset angle formed by the corresponding rulings to $\bar{M}$ and M, namely

$$< \bar{\mathbf{e}}_1, \mathbf{e}_1 >= \cos\psi. \tag{4.1.2}$$

Differentiating this equation, and using equations (2.1.11) and (4.1.1), we can write

$$\bar{p} < \bar{\mathbf{e}}_2, \mathbf{e}_1 > \frac{d\bar{u}}{du} + p < \bar{\mathbf{e}}_1, \mathbf{e}_2 >= -\psi' \sin\psi. \tag{4.1.3}$$

Since $\mathbf{e}_2 = \bar{\mathbf{e}}_2$ at corresponding central points of $\bar{M}$ and M, hence the left-hand side of the previous equation vanishes and we have:

$$\psi' = 0. \tag{4.1.4}$$

This means that ψ is a non-zero constant. Thus, as a result the following Theorem can be given:

Theorem (4.1.1): The offset angle between generating lines of

Bertrand offsets as defined in Definition (4.1.1) remains constant.

In the view of the fact that for a ruled surface and its Bertrand offset the central normals coincide, it follows from the above theorem that the central tangents of the two ruled surfaces also make the same constant angle ψ at the corresponding points on the two striction curves. The relationship between the Blaschke frame of a ruled surface and that of its Bertrand offset can therefore be written as:

$$\begin{bmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \\ \bar{\mathbf{e}}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}. \tag{4.1.5}$$

Note that the above equation is exactly the same as its parallel equation for Bertrand curves, i.e. equation (1.1.6). By taking the derivative of (4.1.5), we obtain

$$\frac{d}{d\bar{u}} \begin{bmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \\ \bar{\mathbf{e}}_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{p} & 0 \\ -\bar{p} & 0 & \bar{q} \\ 0 & -\bar{q} & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \\ \bar{\mathbf{e}}_3 \end{bmatrix}, \tag{4.1.6}$$

where

$$\left.\begin{aligned} \bar{p}\frac{d\bar{u}}{du} &= (p\cos\psi + q\sin\psi), \\ \bar{q}\ \frac{d\bar{u}}{du} &= (-p\sin\psi + q\cos\psi). \end{aligned}\right\} \tag{4.1.7}$$

With aid of geodesic curvature and elimination of $\frac{d\bar{u}}{du}$ between these two equations gives:

$$(\bar{\gamma}-\gamma)\cos\psi+(\bar{\gamma}\gamma+1)\sin\psi=0. \qquad (4.1.8)$$

This is a characterization of Bertrand offsets of ruled surfaces in terms of their spherical curvatures. Therefore, the following theorem may be given:

Theorem (4.1.2): The ruled surfaces $\bar{M}$ and M form a Bertrand offsets if and only if the relationship (4.1.8) hold true.

In particular, we may give the following results:

Result (1): $\psi=0 \Leftrightarrow \bar{\gamma}=\gamma$, i.e., $\bar{M}$ and M are oriented offsets if and only if their spherical curvatures are equals.

Result (2): $\psi=\frac{\pi}{2} \Leftrightarrow \bar{\gamma}\gamma=-1$, i.e., $\bar{M}$ and M are right offsets if and only if their spherical curvatures have opposite sign.

On the other hand, the striction curve $\bar{\mathbf{C}}(u)$ of offset surface $\bar{M}$, in terms of striction curve of base surface M, can therefore be written as:

$$\bar{\mathbf{C}}(u)=\mathbf{C}(u)+\psi^{*}(u)\mathbf{e}_2(u), \qquad (4.1.9)$$

where $\psi^{*}(u)$ denotes the length (the linear offset) of the segment intercepted on the central normal line at corresponding central points of $\bar{M}$ and M. These two striction curve are said to

form a pair of Bertrand mates.

Theorem (4.1.3): The linear offset ψ^* of the common central normal line intercepted by corresponding central points of two Bertrand mates of $\bar{M}$ and M remains constant.

Proof: Since the base curves of $\bar{M}$ and M are their striction curves, we get:

$$< \bar{\mathbf{C}}', \mathbf{C} - \bar{\mathbf{C}} >= 0, \quad < \mathbf{C}', \bar{\mathbf{C}} - \mathbf{C} >= 0. \qquad (4.1.10)$$

Changing the sign of the first left-hand side in equation (4.1.10), and subtracting the remaining one, the equality

$$< \bar{\mathbf{C}}' - \mathbf{C}', \bar{\mathbf{C}} - \mathbf{C} >= 0,$$

is obtained. Integrating this last result yields

$$\|\bar{\mathbf{C}}(u) - \mathbf{C}(u)\|^2 = {\psi^*}^2, \qquad (4.1.11)$$

with ψ^* is constant as stated.

The equation of the offset surface $\bar{M}$, in terms of its base surface M, can therefore be written as:

$$\bar{M}: \bar{\mathbf{L}}(\bar{u}, \mu) = \bar{\mathbf{C}} + \mu \bar{\mathbf{e}}_1,$$

or by equation (4.1.9), we have:

$$\bar{M}: \bar{\mathbf{L}}(\bar{u}, \mu) = \mathbf{C}(u) + \psi^* \mathbf{e}_2 + \mu(\cos\psi\, \mathbf{e}_1 + \sin\psi\, \mathbf{e}_3). \quad (4.1.12)$$

It is clear from the above developments that a ruled surface, in general, has a double infinity of Bertrand offsets. Each Bertrand offset can be generated from equation (4.1.12) using a constant linear offset $\psi^*$and a constant angular offset ψ. Any two ruled surfaces in this family of ruled surfaces are reciprocal of one another in the sense of Definition (4.1.1).

By taking the derivative of (4.1.9) with respect u and applying the Blaschke formulae, we have:

$$\bar{\mathbf{t}}(u) = \{(\cos\sigma - \psi^* p)\mathbf{e}_1 + (\sin\sigma + \psi^* q)\mathbf{e}_3\}\frac{du}{d\bar{u}}, \quad (4.1.13)$$

Also, we have:

$$\bar{\mathbf{t}}(u) = \cos\bar{\sigma}\bar{\mathbf{e}}_1 + \sin\bar{\sigma}\,\bar{\mathbf{e}}_3. \quad (4.1.14)$$

By making use of (4.1.5) and comparing equations (4.1.13) with (4.1.14), it therefore follows that

$$\left.\begin{aligned} &[\cos(\sigma+\psi) - \psi^* \,(p\cos\psi + q\sin\psi)\,\frac{du}{d\bar{u}} = \cos\bar{\sigma}, \\ &[\sin(\sigma+\psi) + \psi^*(-p\sin\psi + q\cos\psi)]\,\frac{du}{d\bar{u}} = \sin\bar{\sigma}. \end{aligned}\right\} \quad (4.1.15)$$

Then, we have

$$\left.\begin{aligned} \frac{du}{d\bar{u}} :&= \frac{\cos\,\bar{\sigma}}{\cos\,(\sigma+\psi) - \psi^*(p\cos\,\psi + q\sin\,\psi)}, \\ &= \frac{\sin\,\bar{\sigma}}{\sin\,(\sigma+\psi) + \psi^*(-p\sin\,\psi + q\cos\,\psi)}. \end{aligned}\right\} \quad (4.1.16)$$

According to the definition of the distribution parameter and geodesic curvature, we have the following expression by (4.1.16):

$$\bar{\delta} = \frac{\sin \bar{\sigma}}{\bar{p}} = \frac{(\delta + \psi^{*}\gamma) \cos \psi + (\Gamma - \psi^{*}) \sin \psi}{\cos \psi + \gamma \sin \psi}, \qquad (4.1.17)$$

is the distribution of the ruled surface $\bar{M}$.

Based on the equations (2.1.2), (2.1.10) and (4.1.12) the unit normal vectors of M and $\bar{M}$ at any point (u, μ) or $(\bar{u}, \mu)$ are respectively, expressed as:

$$\mathbf{N} = \frac{-\delta \mathbf{e}_2 + \mu \mathbf{e}_3}{\sqrt{\delta^2 + \mu^2}}, \quad \bar{\mathbf{N}} = \frac{-\delta \bar{\mathbf{e}}_2 + \mu \bar{\mathbf{e}}_3}{\sqrt{\bar{\delta}^2 + \mu^2}}. \qquad (4.1.18)$$

Equation (4.1.18) states that a ruled surface and its Bertrand offsets are in general, not contact along a common ruling. This is clear the fact that $\mathbf{N}$ and $\bar{\mathbf{N}}$ are not the same. Then question that arises is then under what condition a ruled surface and its Bertrand offsets are contact along a common ruling ?. The contact conditions between M and $\bar{M}$ can be described by the following theorem:

Theorem (4.1.4): Two non-developable Bertrand offsets $\bar{M}$ and M contact along a common ruling if and only if (a) $\delta = \bar{\delta}$, (b) ψ and (c) $\psi^{*} = 0$ or (M and $\bar{M}$ are right helicoids).

Proof: Suppose $\bar{M}$ and M contact along a common ruling

corresponding to u and $\bar{u}$, or $\bar{\mathbf{N}} \times \mathbf{N} = \mathbf{0}$, we have the following expressions by equations (4.1.5) and (4.1.18):

$$\mu\left(\delta^2 \cos\psi - \bar{\delta}\right)\mathbf{e}_1 - \mu^2 \sin\psi\, \mathbf{e}_2 - \mu\delta \sin\psi\, \mathbf{e}_3 = \mathbf{0}. \quad (4.1.19)$$

The above equation must hold true for any value $\mu \neq 0$, which leads to

$$\psi = 0, \quad \text{and} \quad \delta = \bar{\delta}. \quad (4.1.20)$$

By substituting equation (4.1.20) into equation (4.1.17), we get $\psi^*\gamma = 0$. Therefore, either $\psi^* = 0$ or $\gamma = 0$ (M is a right helicoids). The conditions $\psi = 0$ and $\gamma = 0$ substituted in the result (1) gives $\bar{M}$ is a right helicoids too.

Conversely, suppose that three conditions (a), (b) and (c) hold true, then substitute them into the following equation:

$$\mathbf{N} \times \bar{\mathbf{N}} = \frac{-\delta\mathbf{e}_2 + \mu\mathbf{e}_3}{\sqrt{\delta^2 + \mu^2}} \times \frac{-\delta\bar{\mathbf{e}}_2 + \mu\bar{\mathbf{e}}_3}{\sqrt{\bar{\delta}^2 + \mu^2}}. \quad (4.1.21)$$

The result of the above equation is zero vector for any value of μ, which implies that M and $\bar{M}$ are contact to the first order along a common ruling.

Now, assume that M is developable ruled surface ($\delta = 0$). Then, we have:

$$\bar{\delta} = \frac{\psi^* \gamma \cos\psi + (\Gamma - \psi^*) \sin\psi}{\cos\psi + \gamma \sin\psi}. \quad (4.1.22)$$

It is clear from equations (4.1.22), that the ruled surface $\bar{M}$, is not in general, developable ruled surface. Hence, the following corollary can be given:

Corollary (4.1.1): A non developable and developable Bertrand offsets ruled surfaces can not contact along a common ruling to the first order.

From equation (4.1.22), for $\bar{M}$ to be developable ruled surface

$$(1-\psi^*\kappa)\sin\psi+\psi^*\tau\cos\psi=0. \qquad (4.1.23)$$

The above equation must be hold for ψ should be zero, which leads to $\psi^*=0$ which means that the edge of regression is coincident with its offset, i.e. their striction curves are the same. Hence, the following corollary can be given:

Corollary (4.1.2): Two developable Bertrand offsets ruled surfaces contact along a common ruling to the first order if and only if they are oriented offsets.

On the other hand, we can write the reciprocal of (4.1.23) as:

$$(1+\psi^*\bar{\kappa})\sin\bar{\psi}-\psi^*\bar{\tau}\cos\bar{\psi}=0, \qquad (4.1.24)$$

where $\bar{\psi}$ is the analogue of ψ when $\mathbf{C}$ and $\bar{\mathbf{C}}$ are interchanged. In addition, according to $\cos\bar{\psi}=\cos\psi$ and $\sin\bar{\psi}=-\sin\psi$, we find from equations (4.1.23) and (4.1.24) that:

$$\psi^* = \frac{-\sin\psi}{\tau\cos\psi - \kappa\sin\psi} = \frac{\sin\bar{\psi}}{\bar{\tau}\cos\bar{\psi} + \bar{\kappa}\sin\bar{\psi}}. \qquad (4.1.25)$$

Hence, the following corollary can be given:

Corollary (4.1.3): The necessary and sufficient condition for Bertrand offsets $\bar{M}$ and M are developable ruled surfaces is that their striction curves are Bertrand curves.

From equation (4.1.25), developable Bertrand offsets are right offsets if and only if

$$\kappa = \bar{\kappa} = \frac{1}{\psi^*} = constant.$$

This means that their edge of regressions are curves with constant curvatures.

(4.2) The dual formulation

In this subsection the dual vector calculus is used for differential geometry of Bertrand offsets for ruled and developable surfaces. Considering a ruled surface, as a dual curve, then its normal offset is a dual curve having common central normals at all its points with the original base curve. With the dualized form of line representation along with the E. Study's map leads to the following definition for Bertrand offset of a dual curve:

Definition (4.2.1): Two dual curves (ruled surfaces) are said to

be Bertrand offsets of one another if there exists a one-to-one correspondence between their central points such that both curves have a common central normal.

Now, let $\bar{\mathbf{E}}$ be an oriented line attached with the Blaschke frame and has following dual representation:

$$\bar{\mathbf{E}} = X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3, \tag{4.2.1}$$

where $X_i = X_i(u)$, (i =1, 2, 3) are its dual coordinate functions. Then

$$X_1^2 + X_2^2 + X_3^2 = 1. \tag{4.2.2}$$

Differentiating equations (4.2.1) and (4.2.2), and using the Blaschke formulae, we find:

$$\left.\begin{aligned} \bar{\mathbf{E}}' = (X_1' - PX_2)\mathbf{E}_1 + (X_2' + PX_1 - QX_3)\mathbf{E}_2 + (X_3' + QX_2)\mathbf{E}_3, \\ X_1X_1' + X_2X_2' + X_3X_3' = 0. \end{aligned}\right\} \tag{4.2.3}$$

Now, if the dual curves $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\bar{u})$ and $\mathbf{E} = \mathbf{E}(u)$ are Bertrand offsets in the sense of Definition (4.1.1), i.e. $\mathbf{E}_2 = \bar{\mathbf{E}}_2$ (central normal to $\bar{\mathbf{E}}$), then from the first equation of (4.2.3), we have

$$\left.\begin{aligned} (X_1' - PX_2) = 0, \\ (X_2' + PX_1 - QX_3) = < \bar{\mathbf{E}}', \mathbf{E}_2 >, \\ (X_3' + QX_2) = 0. \end{aligned}\right\} \tag{4.2.4}$$

Substituting equations (4.2.4) into the second equation of (4.2.3) and simplifying it, we obtain

$$X_2 = 0. \tag{4.2.5}$$

Making use of this equation into equations (4.2.4), we get

$$\left.\begin{aligned} X_1' &= 0, \\ PX_1 - QX_3 &= <\bar{\mathbf{E}}', \mathbf{E}_2>, \\ X_3' = 0 \Rightarrow X_1 &= \mathrm{C}_1, \\ X_3 &= \mathrm{C}_3 \in D, \end{aligned}\right\} \tag{4.2.6}$$

where C_1 and C_3 are dual constants of integrations. So, we can find a constant dual angle $\Psi = \psi + \varepsilon\psi^*$ such that $\mathrm{C}_1 = \cos\Psi$ and $\mathrm{C}_3 = -\sin\Psi$. Thus we may deduce from equations (4.2.6) that:

$$\left.\begin{aligned} X_1 &= \cos\Psi, \\ X_3 &= -\sin\Psi, \\ P\cos\Psi + Q\sin\Psi &= <\bar{\mathbf{E}}', \mathbf{E}_2>. \end{aligned}\right\} \tag{4.2.7}$$

Then $\Psi = \psi + \varepsilon\psi^*$ is a constant dual angle formed by the generating lines $\bar{\mathbf{E}}$ and $\mathbf{E}$ at corresponding central points of the ruled surfaces $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\bar{u})$ and $\mathbf{E} = \mathbf{E}(u)$. Thus equation (4.2.1) become:

$$\bar{\mathbf{E}}(\bar{u}) = \cos\Psi\,\mathbf{E}_1(u) - \sin\Psi\,\mathbf{E}_3(u). \tag{4.2.8}$$

It is clear from the above developments that a ruled surface, in general, has a dual infinity of Bertrand offsets. Each Bertrand offset can be generated from equation (4.2.8) using a constant linear offset $\psi^* \in \mathbb{R}$ and a constant angular offset $\psi \in [0, 2\pi]$. It should be pointed out that the relationship (4.2.8) is a reciprocal one, i.e. if $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\bar{u})$ is a Bertrand offset of $\mathbf{E} = \mathbf{E}(u)$, then $\mathbf{E} = \mathbf{E}(u)$ is also a Bertrand offset of $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\bar{u})$. Then we may state the following theorem:

Theorem (4.2.1): The offset dual angle between the generating lines of a ruled surface and its Bertrand offset remains constant.

This theorem is a short proof of the two Theorems (4.1.1) and (4.1.3). It is clear that we can find the following:

$$\left.\begin{aligned} \bar{\mathbf{E}} &= \bar{\mathbf{E}}_1(\bar{u}), \\ \bar{\mathbf{E}}_2(\bar{u}) &= \frac{\bar{\mathbf{E}}'}{\|\bar{\mathbf{E}}'\|}, \\ \bar{\mathbf{E}}_3(\bar{u}) &= \bar{\mathbf{E}} \times \frac{\bar{\mathbf{E}}'}{\|\bar{\mathbf{E}}'\|}. \end{aligned}\right\} \tag{4.2.9}$$

as the Blaschke frame of the dual curve $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\bar{u})$. In the view of the fact that for a dual curve and its Bertrand offset the central normals coincide, it follows from the above theorem that the central tangents of the two dual curves also make the same constant dual angle $\Psi = \psi + \varepsilon\psi^*$ at the corresponding points on the two curves. Therefore, we can write that:

$$\begin{bmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{E}}_2 \\ \bar{\mathbf{E}}_3 \end{bmatrix} = \begin{bmatrix} \cos\Psi & 0 & -\sin\Psi \\ 0 & 1 & 0 \\ \sin\Psi & 0 & \cos\Psi \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}. \qquad (4.2.10)$$

Also, note that the above equation is exactly the dual version of equation (1.1.6). If $\bar{u}$ is the arc-length of the striction curve on $\bar{\mathbf{E}}_1 = \bar{\mathbf{E}}_1(\bar{u})$, then differentiating of (4.2.10) with respect to $\bar{u}$ and using (2.3.5), we have:

$$\frac{d}{d\bar{u}} \begin{bmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{E}}_2 \\ \bar{\mathbf{E}}_3 \end{bmatrix} \begin{bmatrix} 0 & \bar{P} & 0 \\ -\bar{P} & 0 & \bar{Q} \\ 0 & -\bar{Q} & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{E}}_2 \\ \bar{\mathbf{E}}_3 \end{bmatrix}, \qquad (4.2.11)$$

where

$$\left.\begin{aligned} \bar{P}\frac{d\bar{u}}{du} &= (P\cos\Psi + Q\sin\Psi), \\ \bar{Q}\frac{d\bar{u}}{du} &= (-P\sin\Psi + Q\cos\Psi). \end{aligned}\right\} \qquad (4.2.12)$$

and consequently

$$\frac{du}{d\bar{u}} = \frac{\bar{P}}{(P\cos\ \Psi + Q\sin\ \Psi)} = \frac{\bar{Q}}{(-P\sin\ \Psi + Q\cos\ \Psi)}. \qquad (4.2.13)$$

This expression may be rewritten in terms of the dual spherical of curvatures to yield

$$(\bar{\Sigma} - \Sigma)\cos\Psi + (\bar{\Sigma}\Sigma + 1)\sin\Psi = 0. \qquad (4.2.14)$$

This is the dual version of equation (4.1.8) and is a new dual

characterization of Bertrand offset of ruled surfaces in terms of their dual spherical curvatures. Therefore, the following theorem may be given:

Theorem (4.2.2): A non-developable ruled surfaces $\mathbf{E}_1 = \mathbf{E}(u)$ and $\bar{\mathbf{E}}_1 = \bar{\mathbf{E}}(\bar{u})$ forms Bertrand offsets ruled surfaces if and only if (4.2.14) hold true.

In terms of Theorem (4.2.2) and formula (4.2.14) the following results can be given:

Result (3): $\Psi = \psi + \varepsilon\psi^* = 0$, i.e. M and $\bar{M}$ their rulings are coincident (or in opposite directions) $\Leftrightarrow \Sigma = \bar{\Sigma}$.

Result (4): $\Psi = \psi + \varepsilon\psi^* = \frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{2}, \psi^* = 0$, i.e. M and $\bar{M}$ are right offsets and their striction curves are identical $\Leftrightarrow$ $\Sigma\bar{\Sigma} \neq -1$.

The real and dual parts of the formula (4.2.14), respectively, are (4.1.8) and

$$\psi^* = \frac{(\bar{\gamma}^* - \gamma^*) \cos\psi + (\bar{\gamma}^*\gamma + \bar{\gamma}\gamma^*) \sin\psi}{(\bar{\gamma} - \gamma) \sin\psi - (1 + \bar{\gamma}\gamma) \cos\psi}. \qquad (4.2.15)$$

The relationship between the dual arc length of $\bar{\mathbf{E}} = \bar{\mathbf{E}}_1(\bar{u})$ and that of its Bertrand offset can be developed as follows: Let $d\bar{\Phi}$ be the dual arc length on the Bertrand offset of the curve $\bar{\mathbf{E}}_1(\bar{u})$, then

$$d\bar{\Phi} = \bar{P}d\bar{u} \Rightarrow d\bar{\Phi} = (\cos\Psi + \Sigma\sin\Psi)d\Phi. \qquad (4.2.16)$$

If we calculate the real and dual parts of (4.2.16), respectively, we have:

$$\left.\begin{aligned} d\bar{\varphi} &= (\cos\psi + \gamma\sin\psi)d\varphi, \\ \bar{\delta} = \frac{d\bar{\varphi}^*}{d\bar{\varphi}} &= \delta + \frac{\gamma^*\sin\psi + \psi^*(\gamma\cos\psi - \sin\psi)}{\cos\psi + \gamma\sin\psi}. \end{aligned}\right\} \quad (4.2.17)$$

The formulas (4.1.8), (4.1.17), (4.2.15) and (4.2.17) unite the dual and scalar formulations of the differential geometry of ruled surfaces. It seems clear that no matter how the Bertrand offset ruled surfaces are represented the equations (4.1.12) or the equivalent (4.2.9), contain the fundamental geometric information describing the shape of the surface.

(4.3) Examples and Remarks

The ruled surface traced by a line fixed in a rigid body undergoing a helical one-parameter spatial motion of constant pitch is fundamental to the curvature theory of ruled surfaces. This surface is generated by the line L carried by a circular helix of radius ρ^* and pitch h. Align the axis of the helix with the z-axis so the equation of the helix is:

$$\boldsymbol{\alpha}(\theta) = (\rho^*\cos\theta, \rho^*\sin\theta, h\theta), \quad \theta \in [0, 2\pi]. \qquad (4.3.1)$$

The direction $\mathbf{e}_1(\theta)$ of the line L lies in the plane normal to the radius vector

$$\mathbf{r}(\theta) = (\cos\theta, \sin\theta, 0), \tag{4.3.2}$$

and at an angle ρ from the z-axis direction $\mathbf{z}$, that is

$$\mathbf{e}_1(\theta) = -\sin\rho\, \mathbf{r}^{\perp} + \cos\rho\, \mathbf{z} = (\sin\rho \sin\theta, -\sin\rho\cos\theta, \cos\rho). \tag{4.3.3}$$

Thus the parametric representation of this ruled surface is

$$\mathbf{L}(\theta, \eta) = \boldsymbol{\alpha}(\theta) + \eta\mathbf{e}_1(\theta); \quad \eta \in \mathbb{R}. \tag{4.3.4}$$

By means of (4.3.1), the unit tangent vector field to helix is:

$$\mathbf{t}(\theta) = (-\rho^* \sin\theta, \rho^* \cos\theta, h); \quad h^2 + {\rho^*}^2 = 1. \tag{4.3.5}$$

It is clear that $\boldsymbol{\alpha}(\theta)$ is the striction curve on the ruled surface $\mathbf{L}(\theta, \eta)$.

The dual vector function representing $\mathbf{L}(\theta, \eta)$ can be expressed as:

$$\mathbf{E}_1(\theta) = (\sin\Lambda \sin\Theta, -\sin\Lambda\cos\Theta, \cos\Lambda), \tag{4.3.6}$$

where

$$\Lambda = \rho + \varepsilon\rho^* = \mathrm{c}_1 + \varepsilon\mathrm{c}_2, \quad \Theta = \theta(1 + \varepsilon h), \tag{4.3.7}$$

and h, c_1 and c_2 are real constants. Thus, it follows that the variational of the Blaschke frame filed is obtained as:

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix} = \begin{bmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \tag{4.3.8}$$

where

$$\left.\begin{aligned} P = p + \varepsilon p^* = (1 + \varepsilon h)\sin\Lambda, \\ Q = (1 + \varepsilon h)\cos\Lambda. \end{aligned}\right\} \tag{4.3.9}$$

Then, the distribution parameter of the ruled surface $\mathbf{L}(\theta, \eta)$ is found as:

$$\delta = h + \rho^* \cot \rho. \tag{4.3.10}$$

According to (2.1.18), one can obtain: It is obvious that the striction curve on the ruled surface $\mathbf{L}(\theta, \eta)$ is a geodesic curve.

Hence, the corresponding equations to (4.2.12) are:

$$\left.\begin{aligned} \bar{P} = (1 + \varepsilon h)\sin\Xi \frac{d\theta}{d\bar{\theta}}, \\ \bar{Q} = (1 + \varepsilon h)\cos\Xi \frac{d\theta}{d\bar{\theta}}, \end{aligned}\right\} \tag{4.3.11}$$

where $\Xi = \Lambda + \Psi$. Therefore the corresponding distribution parameter of the offset ruled surface $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\theta)$ is:

$$\bar{\delta} = \delta - \kappa \cot \rho + (\kappa + \psi^*) \cot(\rho + \psi). \tag{4.3.12}$$

It is clear that the striction curve on the Bertrand offset ruled surface is geodesic curve.

The graphs of the ruled surface $\mathbf{L}(\theta, \eta)$ and its Bertrand offset are shown in Figures (4.3.1) and (4.3.2); the domain is:

$$\theta \in [0, 2\pi],\ \eta \in [-4, 4],\ \rho = \frac{\pi}{4},\ \rho^* = \frac{1}{\sqrt{2}},\ h = \frac{1}{\sqrt{2}}. \quad (4.3.13)$$

Further, the invariants are calculated as:

$$\delta = \frac{7}{12},\ \bar{\delta} = \frac{-5}{12}. \quad (4.3.14)$$

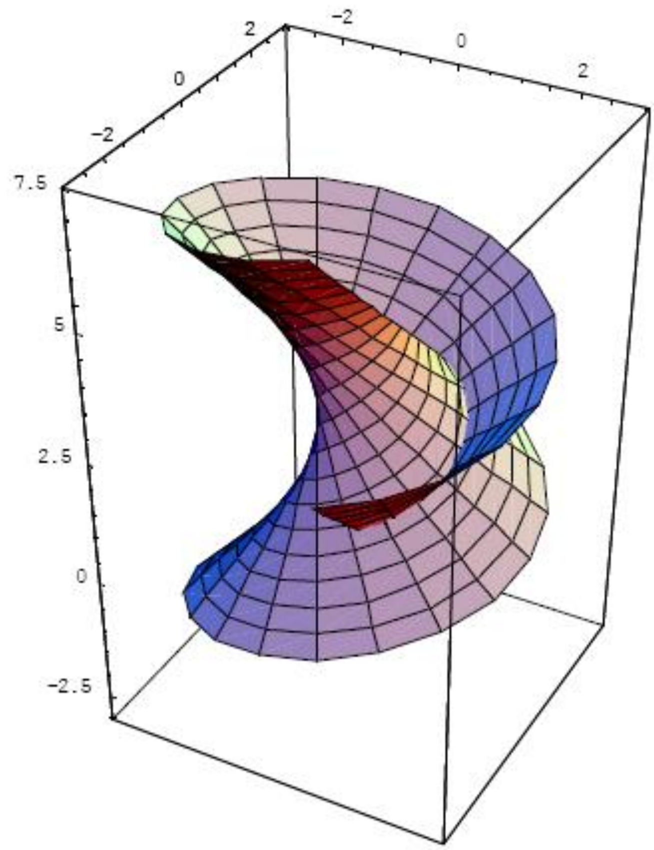

Figure (4.3.1): Ruled surface

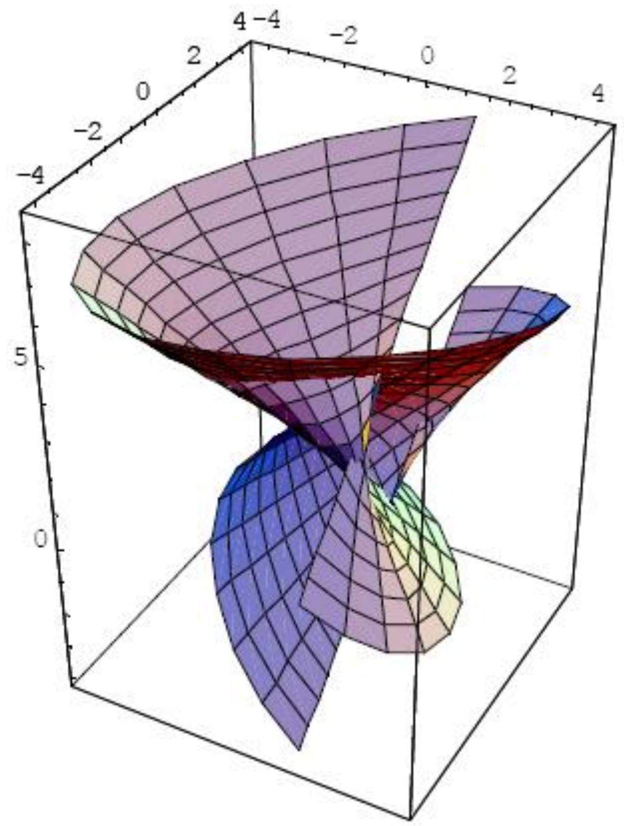

Figure (4.3.2): Bertrand offset

LIST OF ACADEMIC EXPRESSIONS

A	
Addition	إضافة (جمع)
Angle	زاوية
Angular offset	بديل زاوي
Arbitrary	اختياري
Arc-length	طول القوس
Asymptotic strip	شريط تقاربي
B	
Base Curve	منحنى الأساس
Bertrand offsets	بدائل بيرتراند
Binormal Vector	متجه الثانوي
Blaschke frame	اطار بلاشكا
C	
Central angle	الزاوية المركزية
Central normal vector	متجه العمودي المركزي
Central tangent vector	متجه المماس المركزي
Chain rule	قاعدة السلسلة
Circular cylinder	اسطوانة دائرية
Components	مركبات
Constant	ثابت
Continuous	متصلة
Coordinates	إحداثيات
Correspondence	تقابل
Curvature	إنحناء
Curvature strip	شريط إنحناء
Curve	منحنى

Cylindrical	اسطواني
D	
Derivative	مشتقة
Developable	قابل للإنبساط
Differentiable	تفاضلي
Differential geometry	هندسة تفاضلية
Dimension	بعد
Directrix curve	منحنى الدليل
Distance	مسافة
Distribution parameter	بارامتر التوزيع
Division	قسمة
Domain	مجال
Dual	ازدواجي
Dual angle	زاوية ازدواجية
Dual numbers	الأعداد الإزدواجية
Dual unit sphere	كرة الوحدة الإزدواجية
Dual vectors	المتجهات الإزدواجية
E	
Edge of regression	وصلة انحدار
Equality	مساواة
E. Study's map	راسم شتودي
F	
Filed	حقل
Fixed point	نقطة ثابتة
Fixed Space	فضاء ثابت
Frame	اطار
Frenet formulas	صيغ فرينيه
Function	دالة

G	
Geodesic Curvature	الإنحناء الجيوديسي
Geodesic strip	شريط جيوديسي
Generator vector	متجه المولد
H	
Helical motion	حركة حلزونية
Hyperboloid of one-sheet	سطح زائدي ذي طية واحدة
Hyperbolic paraboloid	سطح مكافئ زائد
I	
Independent	مستقل
Indicatrix	الدليل
Infinite	غير محدود
Inner product	ضرب داخلي
Instantaneous	لحظي
Integration	تكاملي
Intersection	تقاطع
Interval	فترة
Invariants	ثوابت- لا متغيرات
L	
Line	خط
Linear offset	بديل خطي
Local	محلي
M	
Moment vector	متجه العزم
Motion	حركة
Moving Space	فضاء متحرك
Multiplication	ضرب
N	

Neighboring generators	المولدات المجاورة
Norm	مقياس (معيار)
Normal	عمودي
Normalized	المعايرة
O	
Offsets	بدائل
One-to-one	احادي التناظر
Opposite sign	اشارة معاكسة
Oriented line	خط موجه
Origin	أصل
Orthogonal	متعامد
Orthogonal matrix	مصفوفة متعامدة
Orthonormal	عياري التعامد
P	
Pair	زوج
Parallel	موازٍ
Parameterization	إعادة تمثيل
Perpendicular	عمودي
Pitch	قفزة
Plane curve	منحنى مستو
Plucker Coordinates	احداثيات بلوكر
Point	نقطة
Position vector	متجه موضع
Positive	موجب
Principal normal	العمودي الرئيسي
Projected angle	زواية الإسقاط
Proper	فعلي
Pure dual	ازدواجي بحت
Pure real	حقيقي بحت

R	
Radius	نصف القطر
Ratio	نسبة
Real-valued	قيمة حقيقية
Regular	منتظم
Representation	تمثيل
Right angle	زواية يمينية
Right helicoid	حلزوني يميني
Ring	حلقة
Rotation	دوران
Ruled Surface	سطح مسطر
Ruling	مولد (مسطر)
S	
Scalar product	الضرب القياسي
Shape	شكل
Skew lines	خطوط متخالفة
Smooth	أملس
Space	فضاء
Spatial motion	حركة عامة
Spherical Curve	المنحنى الكروي
Spherical Curvature	الإنحناء الكروي
Spherical image	الصورة الكروية
Spherical indicatrix	الدليل الكروي
Straight line	خط مستقيم
Strction Curve	منحنى الإنكماش
Strip Curves	شريط المنحنيات
Structural equation	معادلة التكوين (البناء)
Surface	سطح
T	

Tangent Vector	متجه مماس
Taylor expansion	مفكوك تايلور
Three-dimensional	ثلاثي الأبعاد
Torsion	التواء
Tours	
Trajectory	مسار
Trigonometric function	دالة مثلثيه
Triplet	ثلاثي
U	
Unique	وحيد
Unit	وحده
V	
Variable	متغير
Vector	متجه
Vectorial Product	الضرب الإتجاهي
Vertex	رأس

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CONTENTS

CHPTER 4: Bertrand Offsets of Ruled Surface